Series of Functions

In general, we define a series of functions to be any expression of the form

$$\sum_{n=-\infty}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

in which $f_n(x)$, an expression which depends upon n and x, is the n^{th} term of the series. To illustrate this, consider two examples of series of functions,

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+1}{x}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n} \, .$$

In the first case, $f_n(x)$ is given by $\frac{1}{n}\left(\frac{x+1}{x}\right)$. In the second case, $f_n(x)$ is given by $\frac{\sin nx}{n}$. Now, many common types of series of functions exist. These are classified according to the form of f_n. In particular, there are power series,

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n \, ,$$

in which f_n is a power of z or, in greater generality, of $(z-a)$, and in either case $z \in \mathbb{C}$. Dirichlet series have $f_n(z) = a_n/n^z$, $z \in \mathbb{C}$, and are summed over positive values of the index,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \, .$$

Trigonometric series are series of functions of the form $a_n \cos nx + b_n \sin nx$ where, typically, $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx \, .$$

Specific kinds of each of these general types of series of functions are obtained when the coefficients are selected according to a specific formula and somehow correspond to a particular given function $f(x)$, or $f(z)$ in the case $z \in \mathbb{C}$. For example, the Riemann zeta function is a special case of Dirichlet series where the coefficients are all 1,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \, .$$

Fourier series are trigonometric series with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx \, .$$

Now, in the case of power series, three kinds of power series are identified, these are Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n \text{ with } a_n = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{(z-c)^{n+1}}\, dz \, ,$$

Taylor series, which is a special case of Laurent series where the variable z is specialized to the real line and is denoted x to emphasize this, the centre a is real, and the sum is over non-negative values of the index,

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ with } a_n = \frac{f^{(n)}(a)}{n!} \, ,$$

and Maclaurin series, the special case of Taylor series centred at 0, i.e., with $a = 0$,

$$\sum_{n=0}^{\infty} a_n x^n \text{ with } a_n = \frac{f^{(n)}(0)}{n!} \, .$$

Of course, other series are possible. However, this document focuses primarily upon Taylor series and the special case expanded about the origin, Maclaurin series.

Digression on Symmetry

Graphs of functions might exhibit symmetry. Two kinds of symmetry will be important in the development which follows. Study the plots of powers of x in figures 1 and 2 below. What kind of symmetry do you observe? How might you characterize the symmetry in a succinct mathematical expression? What other familiar functions exhibit the same kinds of symmetry?

Reflective symmetry through the origin — Odd Symmetry

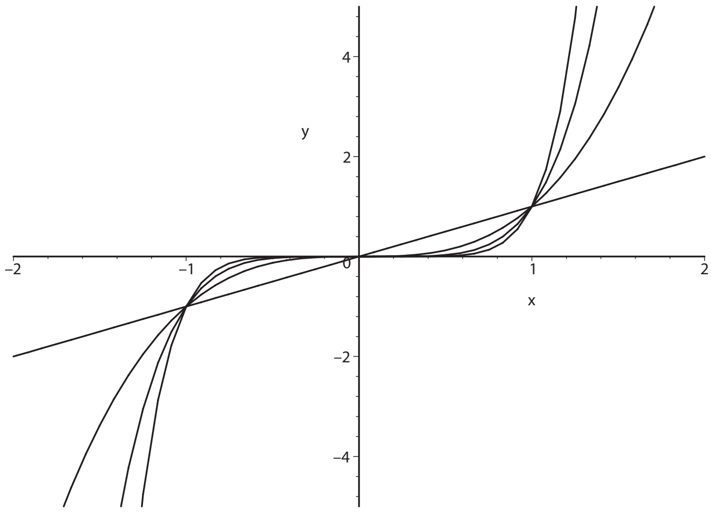

Figure 1: Plot of odd powers of x.

In figure 1, we notice that all of the odd powers of x shown have reflective symmetry through a point, in this case, through the origin. It should be evident that every odd power exhibits this kind of symmetry. Other functions exhibit this kind of symmetry, and when they do, they are said to be “odd functions” because their symmetry is analogous to that of the *odd* powers of x. We may characterize this symmetry algebraically by stating that $f(x)$ is odd if and only if $f(-x) = -f(x)$ for all $x \in \mathcal{D}f$. Why does this make sense? If we locate points on the x-axis symmetrically about the origin, we will notice that the magnitudes of the values of f at these symmetrically disposed points are the same but their signs are opposite.

Reflective symmetry through the y-axis — Even Symmetry

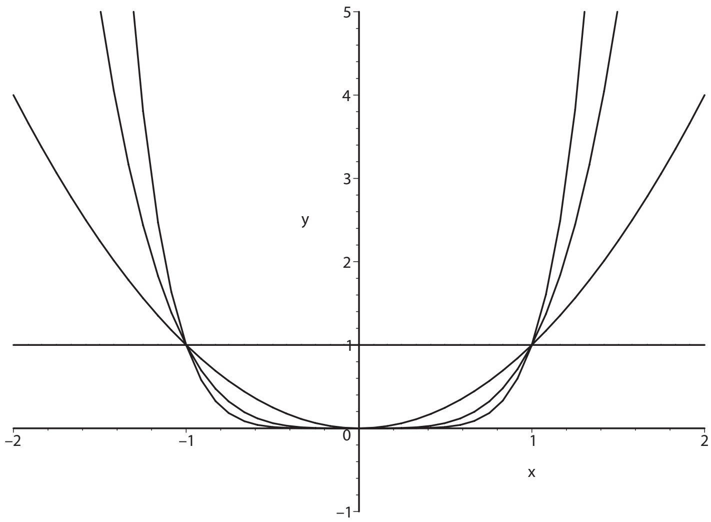

Figure 2: Plot of even powers of x.

Figure 2 displays some even powers of x, and they are seen to have reflective symmetry through an axis, in this case, through the y-axis. It should be evident that every even power exhibits this kind of symmetry. Furthermore, many other functions exhibit symmetry through the y-axis, and when they do, they are said to be "even functions" because their symmetry is the same reflective symmetry through the y-axis as the *even* powers of x. The algebraic characterization of this type of symmetry states that $f(x)$ is even if and only if $f(-x) = f(x)$ for all $x \in \mathcal{D}f$. You must ask yourself why this makes sense. If we locate points on the x-axis symmetrically about the origin, we will notice that the magnitudes of the values of f are the same and their signs are the same as well.

Sine and Cosine

Sine is an odd function, mathematically, $\sin(-\theta) = -\sin\theta$ for all $\theta \in \mathbb{R}$. Cosine is an even function, mathematically, $\cos(-\theta) = \cos\theta$ for all $\theta \in \mathbb{R}$. A quick plot of the graphs of sine and cosine should confirm these claims of symmetry. However, these facts shall subsequently become evident from the discussion of properties of even and odd functions together with our explorations of the Maclaurin series representations of them. Specifically, we will see that the Maclaurin series representation of cosine contains only even powers of x and the Maclaurin series for sine contains only odd powers of x.

Properties of even and odd functions

There are algebraic properties and analytical properties (those having to do with limits, derivatives, and integrals) of even and odd functions which are worth noting. Sums (and differences) of even functions are even, sums (and differences) of odd functions are odd, sums (and differences) with one summand even and one summand odd are neither even nor odd in general. The zero function (the constant function which obtains value 0 for all $x \in \mathbb{R}$) is both even and odd, in fact, it is the only real function of real variables which is both even and odd. Derivatives of even functions are odd, derivatives of odd functions are even. Integrals are more interesting because, while anti-derivatives of odd functions are even, anti-derivatives of even functions are only odd when the constant of integration vanishes. Definite integrals have some nice simplifications however, integrals of odd functions over symmetric intervals about the origin vanish whereas integrals of even functions over symmetric intervals about the origin have a value which is double the integral on either one of the half intervals. We now prove these claims.

Theorem 1 (algebraic properties)**.** *Sums of even functions are even, sums of odd functions are odd, sums of functions where one is even and one is odd are neither even nor odd in general, the zero function is both even and odd.*

Proof. Suppose f and g are even. Consider the function $f + g$. In order to show that $f + g$ is even, we must show that $(f + g)(-x) = (f + g)(x)$ for each x in the domain of $f + g$. That is, we must show that the algebraic characterization holds. We have

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) \quad \text{(by definition of sums of functions)}\\ &= f(x) + g(x) \quad \text{(because f and g are even)}\\ &= (f+g)(x) \quad \text{(by definition of sums of functions)}\end{aligned}$$

Hence, because x was an arbitrarily selected point in the domain, the requirement holds and $f + g$ is even whenever both summands are even. Now we quickly present arguments for the remaining claims. For f and g both odd,

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) \quad \text{(by definition)}\\ &= -f(x) - g(x) \quad \text{(f and g are odd)}\\ &= -(f+g)(x) \quad \text{(by definition)}\end{aligned}$$

Thus, the sum of odd functions is odd. Now, suppose $f(x) = x$ and $g(x) = 1$, then f is odd and g is even, however, the sum $(f + g)(x) = x + 1$ exhibits neither even nor odd symmetry. This counterexample demonstrates that the sum of even and odd functions does not necessarily possess even or odd symmetry. Finally, if a function f is both even and odd, it must satisfy both $f(-x) = f(x)$ and $f(-x) = -f(x)$, hence, equating these, $f(x) = -f(x)$ or $2f(x) = 0$, hence $f(x) = 0$ for all $x \in \mathbb{R}$ as required. □

Some further algebraic properties involving multiplication (and division) and composition follow, but these are not necessarily employed in the discussion on series.

Theorem 2 (further algebraic properties)**.** *Products (and quotients) of even functions are even, products (and quotients) of odd functions are even, the product (or quotient) of an even function with and odd function is odd. Compositions of any function with an even function is even, an even function composed with an even or odd function is even, the composition of two odd functions is odd.*

Proof. In the case of products of two even functions, suppose f and g are both even and consider $(fg)(-x)$.

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x) \,.$$

thus, the product of two even functions is even, and the case for quotients follows similarly. Now suppose that f and g are both odd.

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x) \,,$$

hence, the product of two odd functions is also even. Finally, suppose f is even and g is odd, then

$$(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -(fg)(x) \,,$$

and the product (and similarly the quotient) of an even function with an odd function is odd. Now we prove the claims about compositions, in all cases, suppose the composition $f \circ g$ is defined. Firstly, suppose f is even and g is any function whatsoever and consider $(g \circ f)(-x)$.

$$(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x).$$

Thus, the composition of any function with an even function is even. Now, we demonstrate that the composition of an even function with an odd function is even. Suppose f is even but g is odd.

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x) \,,$$

as required. Finally, to show that the composition of two odd functions is odd, suppose f and g are both odd.

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x) \,,$$

hence, the composition of two odd functions is odd as claimed. □

Theorem 3 (analytical properties)**.** *Derivatives of even functions are odd, derivatives of odd functions are even. For definite integrals, for f even and g odd, we have*

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ and } \int_{-a}^{a} g(x)\,dx = 0 \,.$$

Proof. Suppose f is even and g is odd. Consider $\frac{d}{dx}f(-x)$.

$$\frac{d}{dx}f(-x) = -\frac{d}{dx}f(-x)$$

by chain rule, but, because f is even,

$$\frac{d}{dx}f(-x) = \frac{d}{dx}f(x),$$

and we equate to obtain

$$\frac{d}{dx}f(x) = -\frac{d}{dx}f(-x).$$

Hence, the derivative is odd when the original function is even. Now, consider the derivative of g.

$$\begin{aligned} \frac{d}{dx}g(x) &= -\frac{d}{dx}g(-x) \quad \text{(by symmetry of } g\text{)} \\ &= --\frac{d}{dx}g(-x) \quad \text{(by chain rule)} \\ &= \frac{d}{dx}g(-x) \end{aligned}$$

as required, hence the derivative of g is even whenever g is odd. Now, for definite integrals,

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{(by additivity)} \\
&= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx \quad \text{(by substitution } x = -u \text{ in first integral)} \\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \quad \text{(swapping limits in the first integral)} \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \quad (f(-u) = f(u) \text{ because } f \text{ is even}) \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

becuase the integrals are identical, i.e., the variable of integration is a bound variable and its name doesn't influence the computation. For odd integrands g, we have

$$\begin{aligned}
\int_{-a}^{a} g(x)\,dx &= \int_{-a}^{0} g(x)\,dx + \int_{0}^{a} g(x)\,dx \quad \text{(by additivity)} \\
&= -\int_{a}^{0} g(-u)\,du + \int_{0}^{a} g(x)\,dx \quad \text{(by substitution } x = -u \text{ in first integral)} \\
&= \int_{0}^{a} g(-u)\,du + \int_{0}^{a} g(x)\,dx \quad \text{(swapping limits in the first integral)} \\
&= -\int_{0}^{a} g(u)\,du + \int_{0}^{a} g(x)\,dx \quad (g(-u) = -g(u) \text{ because } g \text{ is odd}) \\
&= 0
\end{aligned}$$

again because the integrals are identical, but this time they are of opposite sign. □

Maclaurin Series

Maclaurin series are a special cases of Taylor series where the expansions are centred at zero. In this section, we develop Maclaurin series of several important elementary functions and manipulate these series to obtain other important series. In fact, in many cases, Maclaurin series may be employed to obtain Taylor series via an appropriate substitution of $(x - a)$.

Maclaurin series are series of functions, in particular, power series corresponding to a given function f, of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n \,.$$

So, to find a Maclaurin series expansion for a particular function f, we may simply compute the derivatives of f, evaluate them at zero, and substitute into this formula. This approach always works whenever a function indeed possesses a power series representation, and it is employed now to find some important series; however, once these series are known, it is possible to employ the known series to find other series without having to go through this tedious process.

Firstly, consider the exponential function.

$$\begin{aligned}
f(x) = e^x &\quad\longrightarrow\quad f(0) = e^0 = 0 \\
f^{(1)}(x) = e^x &\quad\longrightarrow\quad f^{(1)}(0) = e^0 = 0 \\
f^{(2)}(x) = e^x &\quad\longrightarrow\quad f^{(2)}(0) = e^0 = 0 \\
f^{(3)}(x) = e^x &\quad\longrightarrow\quad f^{(3)}(0) = e^0 = 0 \\
\vdots \qquad &\qquad\qquad\quad \vdots \\
f^{(n)}(x) = e^x &\quad\longrightarrow\quad f^{(n)}(0) = e^0 = 0 \,.
\end{aligned}$$

Upon substitution into the expression for the Maclaurin series, we obtain

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \,.$$

Now, consider sine.

$$\begin{aligned} f(x) &= \sin x & \longrightarrow \quad & f(0) = \sin 0 = 0 \\ f^{(1)}(x) &= \cos x & \longrightarrow \quad & f^{(1)}(0) = \cos 0 = 1 \\ f^{(2)}(x) &= -\sin x & \longrightarrow \quad & f^{(2)}(0) = -\sin 0 = 0 \\ f^{(3)}(x) &= -\cos x & \longrightarrow \quad & f^{(3)}(0) = -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & \longrightarrow \quad & f^{(4)}(0) = \sin 0 = 0 \\ & & \vdots \quad & \end{aligned}$$

Observing the cyclic pattern of values 0, 1, 0, -1, 0, etc. and substituting into the expression for the Maclaurin series, we obtain

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} .$$

Now, for cosine, instead of repeating this process, we know that the derivative of sine is cosine, hence, we differentiate the series[1] for sine to obtain the following

$$\cos x = 1 - \frac{1}{3!}3x^2 + \frac{1}{5!}5x^4 - \frac{1}{7!}7x^6 + \frac{1}{9!}9x^8 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n}$$

and, after cancelling within the factorials (observe that $n! = n \times (n-1)!$ and $(2n+1)! = (2n+1) \times (2n)!$), we obtain

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} .$$

Two important series remain to be discussed, and one is a special case of the other, binomial and geometric. I will simply state the expression for the binomial series and use it to derive the expression for the geometric series. As an exercise, try to derive this expression in the manner demonstrated for e^x and sine; this is a useful exercise. The binomial series is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where

$$\binom{k}{n} = \begin{cases} 1 & \text{when n} = 0 \\ \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} & \text{when } n > 0 \end{cases}$$

read "k choose n". As an example of the usage of this strange expression, consider $-1/2$ choose 5.

$$\binom{-1/2}{5} = \frac{(-1/2)(-3/2)(-5/2)(-7/2)(-9/2)}{5!} = -\frac{(1/2)(3/2)(5/2)(7/2)(9/2)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = -\frac{7 \cdot 9}{2^8} .$$

The number of factors in the numerator is n, we begin with the value k and successively subtract 1 until the n^{th} factor is obtained. The denominator simply contains $n!$, pronounced *n factorial* and equal to $n! = n(n-1)! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. Now, we show that the geometric series is a special case of binomial.

$$\frac{1}{1-x} = (1-(-x))^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{-1}{n} (-1)^n x^n$$

[1] And in engineeringland we simply accept that this proposed termwise integration or differentiation is possible, especially in first year when rigorous thought and critical reasoning are beyond our intellectual capacity. Actually, this is dissmissive and inaccurate, it is well within our capacity. The truth is that we don't give a fuck, and this is unfortunate because not giving a fuck can lead to serious squandering of finite resources, financial loss, lawsuits, and revoking of engineering license when our shit fails because we based it upon unproven assumptions. The reality is that differentiation and integration of series of functions entail an interchange of two limiting processes, and this is not always possible. Consider $\lim\limits_{\substack{n\to\infty \\ x\to\infty}} \frac{x}{x+n}$, in which the order of taking limits matters (and hence this stacked limit symbol is not well defined) if we allow x to go to infinity firstly, then the limit is 1, however, if we allow n to go to infinity firstly, the limit is 0. Having said all of this, it turns out that power series possess some nice properties which allow us to differentiate and integrate termwise without worry. For those of you who give a fuck and want to know why, you will be successful in your careers and have mad C.R.E.A.M., please see Walter Rudin's *Principles of Mathematical Analysis* 3rd edition. For those of you who don't give a fuck, don't worry, just integrate and differentiate series of functions termwise, this is suitable for toy problems of little consequence.

now, we need to simplify $\binom{-1}{n}$,

$$\binom{-1}{n}(-1)^n = (-1)^n \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = (-1)^n \frac{(-1)(-2)\cdots(-n)}{n!} = \frac{(1)(2)\cdots(n)}{n!} = \frac{n!}{n!} = 1\,,$$

where, in the third equality from the end, (-1) was pulled from each of the n factors in the numerator, note that $(-1)^n(-1)^n = 1$. Then, upon substitution of this into the series, it simplifies to

$$\sum_{n=0}^{\infty} x^n$$

as expected.

Summary

This document is degenerate in several respects. Examples of Taylor series with centres different from zero have not been discussed. The entire treatment of convergence has been avoided, but the problem set which follows ask questions of convergence. In short, if you don't know how to manipulate radii of convergence of power series, apply the ratio test, if you know how to manipulate the radii, as in integration and differentiation of series do not modify the radii of convergences, composition of series, addition, subtraction, multiplication, and division do modify the radii accordingly, then using known series and their radii is best. The five series which must be memorized are listed here together with their radii of convergence. Finally, this document has not provided a treatment of Taylor's inequality for determining error associated with Taylor polynomials to approximate series. I will try to update this document soon with further discussion of the above topics and full solutions to the practice problems below.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}; \quad R = \infty}$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}; \quad R = \infty}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!} x^{2n}; \quad R = \infty}$$

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n; \quad R = 1}$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n; \quad R = 1}$$

Practice Problems

1. Show the following, and identify the open intervals of convergence where the equalities hold.

   (a) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   (b) $$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

   (c) $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

   (d) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

   (e) $$\frac{x}{x-1} = \sum_{n=0}^{\infty} x^{-n}$$

   (f) $$\frac{x^m}{1-x} = \sum_{n=m}^{\infty} x^n$$

   (g) $$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

   (h) $$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

   (i) $$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

   (j) $$\ln(\cos(x)) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots$$

   (k) $$\frac{e^x}{\cos(x)} = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \cdots$$

   (l) $$(1+x)e^x = \sum_{n=1}^{\infty} \frac{n+1}{n!} x^n$$

2. Find the Taylor series of the given function f about the given centre a. State the radius of convergence.

   (a) $$f(x) = \frac{1}{3x+2};\ a = 0$$

   (b) $$f(x) = \frac{1}{4+x^2};\ a = 0$$

   (c) $$f(x) = \frac{1}{x+3};\ a = 2$$

   (d) $$f(x) = \cos(x^2);\ a = 0$$

   (e) $$f(x) = \frac{1}{\sqrt{1+x}};\ a = 0$$

   (f) $$f(x) = e^{5x};\ a = 0$$

   (g) $$f(x) = \ln(1+2x);\ a = 0$$

   (h) $$f(x) = (1+3x)^{3/2};\ a = 0$$

   (i) $$f(x) = x^4 + 3x^2 - 2x + 1;\ a = 0$$

   (j) $$f(x) = x^4 + 3x^2 - 2x + 1;\ a = 1$$

   (k) $$f(x) = \frac{1}{(x+2)^3};\ a = 0$$

   (l) $$f(x) = \frac{1}{x^2+8x+15};\ a = 0$$

   (m) $$f(x) = x^x;\ a = 3$$

   (n) $$f(x) = \arctan(x);\ a = 0$$

   (o) $$f(x) = \sqrt{x+3};\ a = 0$$

   (p) $$f(x) = \frac{x^2}{(1+x^2)^2};\ a = 0$$

   (q) $$f(x) = x(1-x)^{1/3};\ a = 0$$

   (r) $$f(x) = \frac{1}{3x+2};\ a = 0$$

   (s) $$f(x) = \arcsin(x^2);\ a = 0$$

   (t) $$f(x) = \ln \frac{1+x/\sqrt{2}}{1-x/\sqrt{2}};\ a = 0$$

(u)
$$f(x) = \operatorname{sinc} x = \frac{\sin x}{x};\ a = 0$$

(v) the error function
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt;\ a = 0$$

(w) the Fresne cosinel integral
$$\mathrm{C}(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt;\ a = 0$$

(x) the Fresnel sine integral
$$\mathrm{S}(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt;\ a = 0$$

3. Find the function to which the given series converges, and state the radius of convergence.

(a)
$$\sum_{n=1}^{\infty} n x^{n-1}$$

(b)
$$\sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

(c)
$$\sum_{n=1}^{\infty} (n+1) x^{n-1}$$

(d)
$$\sum_{n=1}^{\infty} n^2 x^{n-1}$$

(e)
$$\sum_{n=1}^{\infty} (n^2 + 2n) x^n$$

(f)
$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^n$$

(g)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

(h)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{2n}$$

(i)
$$\sum_{n=2}^{\infty} n 3^n x^{2n}$$

(j)
$$\sum_{n=0}^{\infty} \frac{n+1}{n+2} x^n$$

4. Show that the given series converges to the given value.

(a)
$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$$

(b)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin 1$$

(c)
$$\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} = \cos 3$$

(d)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e} - 1$$

(e)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n}(2n+1)!} = 3 \sin \frac{1}{3} - 1$$

(f)
$$\sum_{n=1}^{\infty} \frac{2^n}{n 3^n} = \ln 3$$

(g)
$$\sum_{n=1}^{\infty} \frac{1}{n 2^n} = \ln 2$$

(h)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}} = -\frac{1}{5}$$

(i)
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

(j)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

(k)
$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{3^{2n}} = -\frac{9}{100}$$

5. Approximate to three decimal places. In all cases, the resulting anti-derivative is alternating, and, consequently, the alternating series error bound may be employed. There are many cases where the resulting series is not alternating and Taylor's inequality must be employed together with the integral form of the triangle inequality. These cases are not covered in these exercises, but must be learned.

(a)
$$\int_0^1 \operatorname{sinc} x \, dx = \int_0^1 \frac{\sin x}{x} \, dx$$

(b)
$$\int_0^{1/2} \cos x^2 \, dx$$

(c)
$$\int_0^{2/3} \frac{1}{x^4+1} \, dx$$

(d)
$$\int_0^{1/2} \frac{1}{1+x^3} \, dx$$

(e)
$$\int_0^{0.3} e^{-x^2} \, dx$$

6. Evaluate the following limits using series

(a)
$$\lim_{x \to 0} \frac{\tan x}{x}$$

(b)
$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

(c)
$$\lim_{x \to 0} \frac{(1-\cos x)^2}{3x^4}$$

(d)
$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$$

(e)
$$\lim_{x \to \infty} x \sin \frac{1}{x}$$

(f)
$$\lim_{x \to 0} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x} \right)$$

7. Employ series to solve the following differential equations.

(a)
$$y' + 3y = 4$$

(b)
$$y'' + y' = 0$$

(c)
$$xy' - 4y = 3x$$

(d)
$$4xy'' + 2y' + y = 0$$

(e)
$$y'' + y = 0$$

(f)
$$xy'' + y = 0$$

Solutions

It took me about three hours and a large coffee to generate this solution set (which I will grant is partially, albeit deliberately, incomplete, and, furthermore, took about ten hours to typeset). My recommendation is that you become this quick with an error rate of zero. In engineering, you want the mathematics to be second nature because the interesting stuff is the physics and the application of mathematics to solve real engineering problems. You don't want to always be looking things up online, in a book, or in some external source, you want these techniques to be in the fore of your mind so you can solve real problems easily and effectively.

1. (a) We have already demonstrated that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The point of the question is to ensure that you know how to obtain this result. If you have any doubt, reread the derivation. Now, for the interval of convergence, apply the ratio test. In other words, we must find values of x for which

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1\,.$$

The n^{th} term is $x^n/n!$, hence, we consider

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|\,.$$

Begin by collapsing the absolute value to only apply to the part which has a potential to be negative via the rules $|xy| = |x|\,|y|$, $|x/y| = |x|/|y|$, $|-x| = |x|$, and, the triangle inequality $|x+y| \leq |x| + |y|$ (not needed here). Thus,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \frac{n!}{(n+1)!}$$

upon grouping similar expressions together. Then, cancelling, we obtain

$$\lim_{n\to\infty} \frac{|x| n!}{(n+1)n!} = |x| \lim_{n\to\infty} \frac{1}{(n+1)}$$

and this converges to 0 for all $x \in \mathbb{R}$ and $0 < 1$, hence, the given series converges absolutely for all $x \in \mathbb{R}$ (i.e., the interval of convergence is $\mathbb{R}$ or (∞, ∞)) and the radius is $R = \infty$ because the radius is half of the width of the interval of convergence.

(b) The question one must ask oneself is "what does $\ln(1-x)$ look like?" So, amongst the five memorized expressions, the derivative of $\ln(1-x)$ looks awefully similar to a geometric series. Thus, we have

$$\frac{d}{dx} \ln(1-x) = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n\,.$$

However, and this is really important, this is the series representation of the derivative of the desired function, we must effectively solve the separable differential equation

$$\frac{d}{dx} \ln(1-x) = -\sum_{n=0}^{\infty} x^n\,.$$

We integrate both sides, and termwise integration on the right yields

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C\,.$$

Upon substitution of $x = 0$, it is evident that C must vanish, hence

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\,.$$

Now, we may change the variable of summation; increase the starting index by 1 and decrease the index within the general term to compensate for the shift, hence

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \,.$$

To obtain the radius of convergence, we could apply the ratio test as in part (a), but it is more expedient to observe that geometric series converges when $|x| < 1$, i.e., $R = 1$, and that integration does not alter the radius, hence, the radius of the resulting series is $R = 1$. Thus, the interval of convergence is $(-1, 1)$.

(c) This series is easily obtained from 1 (b) upon substitution of $-x$ for x. The radius remains $R = 1$ because the composition with $-x$ simply reflects the interval of convergence but does not scale it. Hence, the open interval of convergence is $(-1, 1)$.

(d) $\sum x^n$ is geometric, hence $|x^{n+1}|/|x^n| = |x|$, the common ratio, and we know that, for the convergence of infinite geometric series, the common ratio must satisfy $|x| < 1$. Hence, $R = 1$ and the open interval of convergence is $(-1, 1)$.

(e)

$$\frac{x}{x-1} = \frac{1}{1-frac1x} = \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n = \sum_{n=0}^{\infty} x^{-n}$$

this is a substitution within a geometric series as well. $\left|\frac{1}{x}\right| < 1$ or $1 < |x|$. This seems to be a pathalogical example because the interval of convergence appears to be outside of the unit disc centred at the origin. I will allow you to think about it.

(f)

$$\frac{x^m}{1-x} = x^m \frac{1}{1-x} = x^m \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+m} \,.$$

Upon change of variable in the index of summation, by starting the series at initial index m, we compensate for the shift by reducing all of the instances of the index within the general term by m and obtain

$$\frac{x^m}{1-x} = \sum_{n=m}^{\infty} x^n \,.$$

(g) Think about derivatives of geometric series.

(h)

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}}{2} = \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \frac{x^n}{n!}$$

but

$$1-(-1)^n = \begin{cases} 0 & \text{when } n \text{ even} \\ 2 & \text{when } n \text{ odd} \end{cases}$$

hence

$$\sinh x = \sum_{\substack{=0 \\ n \text{ odd}}}^{\infty} \frac{x^n}{n!} \,.$$

If we apply a change of variable in the index of summation, that is, if we allow $n = 2k+1$ for $k = 0, 1, 2, \ldots$, which is odd for all such k, we may write this equivalently as

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \,.$$

The radius is $R = \infty$ because each of the series which are combined have infinite radius.

Now, an alternative solution, which I belive is superior, relies on the fact that

$$\boxed{i \sinh x = \sin ix}$$

Thus, we substitute ix into the series representation of sine and divide by i to obtain

$$\sinh x = \frac{1}{i}\sin ix = \frac{1}{i}\sum_{n=0}^{\infty}(-1)^n\frac{(ix)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(i)^{2n}x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

as expected.

(i)

$$\boxed{\cosh x = \cos ix}$$

Thus,

$$\cosh x = \cos ix = \sum_{n=0}^{\infty}(-1)^n\frac{(ix)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(i)^{2n}x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

(j) Carry out the composition with care...

(k) Carry out the composition with care...

(l)

$$\begin{aligned}(1+x)e^x &= (1+x)\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} + x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} + \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n(n-1)!} + \frac{1}{(n-1)!}\right)x^n = 1 + \sum_{n=1}^{\infty}\left(\frac{1+n}{n(n-1)!}\right)x^n\\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1+n}{n!}\right)x^n = \sum_{n=0}^{\infty}\frac{n+1}{n!}x^n\,.\end{aligned}$$

2. (a)

$$\frac{1}{3x+2} = \frac{1}{2\left(1-\left(\frac{-3x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-3x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n3^n}{2^{n+1}}x^n$$

For convergence, we must have $\left|\frac{-3x}{2}\right| < 1$ or $|x| < \frac{2}{3}$, thus $R = \frac{2}{3}$.

(b)

$$\frac{1}{4+x^2} = \frac{1}{4\left(1-\left(-\left(\frac{x}{2}\right)^2\right)\right)} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\left(\frac{x}{2}\right)^2\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{4^{n+1}}$$

For convergence, we must have $\left|-\left(\frac{x}{2}\right)^2\right| < 1$ or $|x| < 2$, thus $R = 2$.

(c)

$$\frac{1}{x+3} = \frac{1}{x-2+5} = \frac{1}{5\left(1-\left(-\frac{(x-2)}{5}\right)\right)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{(x-2)}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{5^{n+1}}$$

For convergence, $\left|-\frac{(x-2)}{5}\right| < 1$ or $|x-2| < 5$, thus $R = 5$.

(d)

$$\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}\,; \quad R = \infty$$

(e)

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^n$$

but

$$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)\cdots(-1/2-n+1)}{n!} = \frac{(-1/2)(-3/2)\cdots(-n+1/2)}{n!}$$
$$= (-1)^n \frac{(1/2)(3/2)\cdots(n-1/2)}{n!} = (-1)^n \frac{(1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}$$
$$= (-1)^n \frac{(2n)!}{2^n\, n!\, 2^n\, n!} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

Thus,

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} x^n\,; \quad R = 1$$

(f)

$$e^{5x} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n\,; \quad R = \infty$$

(g) ...

(h) ...

(i) $x^4 + 3x^2 - 2x + 1$ is already a Maclaurin series with $R = \infty$

(j) There are two approaches to this problem. We may employ the definition of Taylor series, take derivatives, evaluate them at $x = 1$, and substitute into the formula. The second approach involves algebraic transformation of the polynomial to be expanded in powers of $(x-1)$ instead of powers of x. We begin with the first approach.

$$\begin{aligned} f(x) &= x^4 + 3x^2 - 2x + 1 &\longrightarrow\quad & f(1) = 3 \\ f^{(1)}(x) &= 4x^3 + 6x - 2 &\longrightarrow\quad & f^{(1)}(1) = 8 \\ f^{(2)}(x) &= 12x^2 + 6 &\longrightarrow\quad & f^{(2)}(1) = 18 \\ f^{(3)}(x) &= 24x &\longrightarrow\quad & f^{(3)}(1) = 24 \\ f^{(4)}(x) &= 24 &\longrightarrow\quad & f^{(4)}(1) = 24 \end{aligned}$$

and all subsequent derivatives vanish, hence

$$x^4 + 3x^2 - 2x + 1 = 3 + 8(x-1) + \frac{18}{2}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$
$$= 3 + 8(x-1) + 9(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

and it obviously converges everywhere. Now, the second approach is algebraic. We seek an expansion in terms of $(x-1)$ because the result must be a Taylor series about the given centre $x = 1$. Thus, we begin by rewriting $(x-1)^4$ in terms of x^4 and lower powers, substituting the result into the polynomial, and repeating for $(x-1)^3$, etc.. Now,

$$(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1 \quad \text{or} \quad x^4 = (x-1)^4 + 4x^3 - 6x^2 + 4x - 1$$

substituting, we obtain

$$f(x) = [(x-1)^4 + 4x^3 - 6x^2 + 4x - 1] + 3x^2 - 2x + 1 = (x-1)^4 + 4x^3 - 3x^2 + 2x$$

Now,

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1 \quad \text{or} \quad x^3 = (x-1)^3 + 3x^2 - 3x + 1$$

substituting, we obtain

$$f(x) = (x-1)^4 + 4[(x-1)^3 + 3x^2 - 3x + 1] - 3x^2 + 2x = (x-1)^4 + 4(x-1)^3 + 9x^2 - 10x + 4$$

Now,

$$(x-1)^2 = x^2 - 2x + 1 \quad \text{or} \quad x^2 = (x-1)^2 + 2x - 1$$

substituting, we obtain

$$f(x) = (x-1)^4 + 4(x-1)^3 + 9[(x-1)^2 + 2x - 1] - 10x + 4 = (x-1)^4 + 4(x-1)^3 + 9(x-1)^2 + 8x - 5$$

Finally,

$$(x-1) = x-1 \quad \text{or} \quad x = (x-1)+1$$

substituting, we obtain

$$f(x) = (x-1)^4+4(x-1)^3+9(x-1)^2+8[(x-1)+1]-5 = (x-1)^4+4(x-1)^3+9(x-1)^2+8(x-1)+3$$

as expected.

(k)

$$\frac{1}{(x+3)^3} = \frac{1}{27}\frac{1}{\left(1+\frac{x}{3}\right)^3} = \frac{1}{27}\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{27}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\binom{-3}{n}\frac{x^n}{3^{n+3}}$$

now,

$$\begin{aligned}\binom{-3}{n} &= \frac{(-3)(-4)(-5)\cdots(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdots(-n-2)}{n!}\\ &= (-1)^n\frac{3\cdot4\cdot5\cdots(n+2)}{n!} = (-1)^n\frac{2\cdot3\cdot4\cdot5\cdots(n+2)}{2\,n!}\\ &= (-1)^n\frac{n!(n+1)(n+2)}{2\,n!} = \frac{(-1)^n(n+1)(n+2)}{2}\end{aligned}$$

hence,

$$\frac{1}{(x+3)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2\cdot3^{n+3}}x^n$$

An alternative solution begins with

$$\frac{1}{x+3} = \frac{1}{3\left(1-\left(-\frac{x}{3}\right)\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}x^n$$

and differentiating twice,

$$\begin{aligned}\frac{1}{x+3} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}x^n\\ -\frac{1}{(x+3)^2} &= \sum_{n=0}^{\infty}(-1)^n\frac{n}{3^{n+1}}x^{n-1}\\ \frac{2}{(x+3)^3} &= \sum_{n=0}^{\infty}(-1)^n\frac{n(n-1)}{3^{n+1}}x^{n-2} = \sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{3^{n+1}}x^{n-2}\end{aligned}$$

so, by a shift of index,

$$\frac{1}{(x+3)^3} = \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2\cdot3^{n+3}}x^n$$

as expected.

(l) ...

(m) ...

(n)

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^1} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

integrating, we obtain

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + C$$

and, upon substitution of $x=0$, C must vanish, hence

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

3. (a) The series looks like the derivative of the geometric series. Differentiate both sides of

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

to obtain

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

and observe that, when the index $n = 0$, the contribution to the series is 0, hence, we may begin the series at $n = 1$ with no loss of information. Thus,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}\,.$$

$R = 1$ because differentiation does not alter the radius of convergence.

(b)

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{2}{(1-x)^3} &= \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}
\end{aligned}$$

and we see that $\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ converges to $\frac{2}{(1-x)^3}$ on the interval $(-1, 1)$, thus $R = 1$.

(c)

$$\begin{aligned}
\sum_{n=1}^{\infty}(n+1)x^{n-1} &= \sum_{n=1}^{\infty} nx^{n-1} + \sum_{n=1}^{\infty} x^{n-1} \\
&= \sum_{n=1}^{\infty} nx^{n-1} + \sum_{n=0}^{\infty} x^{n}
\end{aligned}$$

where the first term is the derivative of the geometric series and the second term is the geometric series. Thus

$$\sum_{n=1}^{\infty}(n+1)x^{n-1} = \frac{1}{(1-x)^2} + \frac{1}{1-x} = \frac{1}{(1-x)^2} + \frac{1-x}{(1-x)^2} = \frac{2-x}{(1-x)^2}$$

and, again, $R = 1$.

(d) Differentiating the geometric series and multiplying by x, obtain

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{x}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n}
\end{aligned}$$

now, differentiating a second time, we obtain

$$\begin{aligned}
\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} &= \sum_{n=1}^{\infty} n^2x^{n-1} \\
\frac{1-x^2}{(1-x)^4} &= \sum_{n=1}^{\infty} n^2x^{n-1} \\
\frac{1-x^2}{(1-x)^4} &= \sum_{n=1}^{\infty} n^2x^{n-1}
\end{aligned}$$

with $R = 1$.

(e)

$$\sum_{n=1}^{\infty} \left(n^2+2n\right) x^n = \sum_{n=1}^{\infty} n^2 x^n + \sum_{n=1}^{\infty} 2n x^n \,.$$

From part (d), we have

$$\frac{1-x^2}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

and, multiplying through by x,

$$\frac{x\left(1-x^2\right)}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^n \,.$$

From part (a), we have

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

and, multiplying through by $2x$,

$$\sum_{n=1}^{\infty} 2n x^n = \frac{2x}{(1-x)^2} \,.$$

Upon substitution of these results into the decomposition of the given series shown above, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(n^2+2n\right) x^n &= \frac{2x}{(1-x)^2} + \frac{x\left(1-x^2\right)}{(1-x)^4} \\
&= x(1-x^2)\left(\frac{2x}{(1-x)^4} + \frac{x\left(1-x^2\right)}{(1-x)^4}\right) \\
&= \frac{x(1-x^2)(1+2x)}{(1-x)^4} \,.
\end{aligned}$$

4. (a) We know that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \,.$$

Hence, with $x=2$,

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 \,.$$

(b) We know that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x \,.$$

Hence, with $x=1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin 1 \,.$$

(c) We know that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x \,.$$

Hence, with $x=3$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} = \cos 3 \,,$$

but, $3^2=9$, so

$$\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} = \cos 3 \,.$$

(d) We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\,,$$

so,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}\,.$$

Hence, with $x = -1$,

$$e^{-1} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,,$$

or

$$\frac{1}{e} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\,.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin 1\,.$$

(e) We know

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,,$$

thus

$$\sin x - x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,.$$

Dividing by x,

$$\frac{\sin x - x}{x} = \frac{\sin x}{x} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,.$$

Hence, with $x = 1/3$,

$$\frac{\sin \frac{1}{3}}{\frac{1}{3}} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{3}\right)^{2n+1}\,,$$

or

$$3 \sin \frac{1}{3} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{3}\right)^{2n+1}\,.$$

as required.

(f)

$$\sum_{n=1}^{\infty} \frac{2^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{3}\right)^n\,.$$

So we seek the function to which

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges and substitute $2/3$ for x. Integrating both sides of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

yields

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C\,,$$

and, when $x = 0$, the constant must vanish, hence

$$\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,.$$

Now, substitution of 2/3 (which is in the interval of convergence of the power series as it must be) yields

$$\ln\left|1-\frac{2}{3}\right| = -\sum_{n=1}^{\infty}\frac{2^n}{n3^n}\,,$$

thus

$$\ln 3 = \sum_{n=1}^{\infty}\frac{2^n}{n3^n}\,.$$

(g) Same process as part (f).

(h)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1+\sum_{n=1}^{\infty}x^n$$

so

$$\sum_{n=1}^{\infty}x^n = \frac{1}{1-x}-1 = \frac{1-(1-x)}{1-x} = \frac{x}{1-x}\,,$$

and, with $x=-1/2^2$,

$$\begin{aligned}\frac{-1/4}{1-(-1/4)} &= \sum_{n=1}^{\infty}\left(\frac{-1}{2^2}\right)^n\\ \frac{-1/4}{-5/4} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}}\\ -\frac{1}{5} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}}\end{aligned}$$

as required.

(i) We have seen (i.e. you should remember the solutions to problems you've solved in order to become an effective problem solver) that

$$\sum_{n=1}^{\infty}nx^{n-1} = \frac{1}{(1-x)^2}\,.$$

Multiplying through by x, obtain

$$\sum_{n=1}^{\infty}nx^n = \frac{x}{(1-x)^2}\,.$$

With $x=1/2$, obtain

$$\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{(1/2)^2} = \frac{1/2}{1/4} = 2$$

as required.

(j) Consider the series for arctan evaluated at 1.

(k) Same process as in part (i).

5. (a)

$$\begin{aligned}\int_0^1\frac{\sin x}{x}\,dx &= \int_0^1\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}\,dx\\ &= \int_0^1\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n+1)!}\,dx\\ &= \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)(2n+1)!}\Bigg|_0^1\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}\end{aligned}$$

Now, the series is alternating, so we employ the alternating series erorr bound, that $|S - S_N| \leq |a_{N+1}|$, i.e., that the true error associated with truncating an alternating series at index N is the magnitude of the next term. As always, set the error bound to be less than or equal to the desired error because this ensures that the true error will be less than the desired error because the true error is less than the error bound, and obtain, in this case,

$$\frac{1}{(2N+3)(2N+3)!} \leq 0.0005$$

because p decimals of accuracy entails an error of $5 \times 10^{-(p+1)}$. Thus,

$$\begin{aligned}\frac{1}{(2N+3)(2N+3)!} &\leq \frac{5}{10000}\\ \frac{100 \cdot 100}{5} &\leq (2N+3)(2N+3)!\\ 2000 &\leq (2N+3)(2N+3)!\end{aligned}$$

and the smallest natural number N which satisfies this is $N = 2$. Thus

$$\int_0^1 \frac{\sin x}{x}\, dx \approx \sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

with at least 3 decimals of accuracy.

(b)

$$\begin{aligned}\int_0^{1/2} \cos x^2\, dx &= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}\, dx\\ &= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\, dx\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\Bigg|_0^{1/2}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n+1}(4n+1)(2n)!}\end{aligned}$$

and this series is alternating, so we may employ the alternating series error bound as before.

$$\begin{aligned}\left|\frac{(-1)^{(N+1)}}{2^{4(N+1)+1}(4(N+1)+1)(2(N+1))!}\right| &\leq 0.0005\\ \frac{1}{2^{4N+5}(4N+5)(2N+2)!} &\leq \frac{5}{10000}\\ \frac{10000}{5} &\leq 2^{4N+5}(4N+5)(2N+2)!\\ 2000 &\leq 2^{4N+5}(4N+5)(2N+2)!\end{aligned}$$

and the smallest natural number N which satisfies this is $N = 1$. Thus

$$\int_0^{1/2} \cos x^2\, dx \approx \sum_{n=0}^{1} \frac{(-1)^n}{2^{4n+1}(4n+1)(2n)!}$$

with at least three decimals of accuracy.

(c)

$$
\begin{aligned}
\int_0^{2/3} \frac{1}{x^4+1}\,dx &= \int_0^{2/3} \frac{1}{1-(-x^4)}\,dx \\
&= \int_0^{2/3} \sum_{n=0}^{\infty} (-x^4)^n\,dx = \int_0^{2/3} \sum_{n=0}^{\infty} (-1)^n x^{4n}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}\Bigg|_0^{2/3} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n+1}}{3^{4n+1}(4n+1)}
\end{aligned}
$$

the substitution of 2/3 into the series is possible because this value falls into the interval of convergence of the series, otherwise, this series would not be usable to evaluate the definite integral. Now, the series is alternating, thus, apply the alternating series error bound as before.

$$
\begin{aligned}
\left|\frac{(-1)^{N+1}2^{4(N+1)+1}}{3^{4(N+1)+1}(4(N+1)+1)}\right| &\le 0.0005 \\
\frac{2^{4N+5}}{3^{4N+5}(4N+5)} &\le \frac{5}{10000} \\
\frac{10000}{5} &\le \frac{3^{4N+5}(4N+5)}{2^{4N+5}} \\
\frac{2^5}{3^5}\cdot 2000 &\le \frac{81^N(4N+5)}{16^N}
\end{aligned}
$$

and $N=2$ is the smallest integer for which this inequality holds. Thus

$$
\int_0^{2/3} \frac{1}{x^4+1}\,dx \approx \sum_{n=0}^{2} \frac{(-1)^n 2^{4n+1}}{3^{4n+1}(4n+1)}\,.
$$

6. (a)

$$
\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}
$$

thus

$$
\frac{\tan x}{x} = \frac{1 - \frac{x^2}{3!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}
$$

and

$$
\lim_{x\to 0} \frac{\tan x}{x} = \frac{1 - \frac{x^2}{3!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} = 1\,.
$$

(b)

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{x^2} = \lim_{x\to 0} \frac{1}{2!} - \frac{x^2}{4!} + \cdots \\
&= \frac{1}{2}
\end{aligned}
$$

(c)

$$\begin{aligned}
\lim_{x\to 0} \frac{(1-\cos x)^2}{3x^4} &= \lim_{x\to 0} \frac{\left(1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)\right)^2}{3x^4} \\
&= \lim_{x\to 0} \frac{\left(\frac{x^2}{2}-\frac{x^4}{4!}+\cdots\right)^2}{3x^4} \\
&= \lim_{x\to 0} \frac{x^4\left(\frac{1}{2}-\frac{x^2}{4!}+\cdots\right)^2}{3x^4} \\
&= \frac{1}{3}\lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{4!}+\cdots\right)^2 = \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2 \\
&= \frac{1}{12}
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty}\binom{1/2}{n}x^n - 1}{x} \\
&= \lim_{x\to 0} \frac{\left(\binom{1/2}{0}+\binom{1/2}{1}x+\binom{1/2}{2}x^2+\cdots\right)-1}{x} \\
&= \lim_{x\to 0} \frac{\binom{1/2}{1}x+\binom{1/2}{2}x^2+\cdots}{x} \\
&= \lim_{x\to 0} \binom{1/2}{1}+\binom{1/2}{2}x+\cdots \\
&= \frac{1}{2}
\end{aligned}$$

(e)

$$\begin{aligned}
\lim_{x\to\infty} x\sin\frac{1}{x} &= \lim_{x\to\infty} x\left(\frac{1}{x}-\frac{\left(\frac{1}{x}\right)^3}{3!}+\frac{\left(\frac{1}{x}\right)^5}{5!}+\cdots\right) \\
&= \lim_{x\to\infty} 1-\frac{\left(\frac{1}{x}\right)^2}{3!}+\frac{\left(\frac{1}{x}\right)^4}{5!}+\cdots \\
&= 1
\end{aligned}$$

(f) (Incidentally, the first term is hyperbolic cotangent.) Anyway, proceed as above.

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}-\frac{1}{x}\right) \\
&= \lim_{x\to 0}\left(\frac{\left(1+x+x^2/2+x^3/3!+\cdots\right)+\left(1-x+x^2/2-x^3/3!+\cdots\right)}{\left(1+x+x^2/2+x^3/3!+\cdots\right)-\left(1-x+x^2/2+x^3/3!+\cdots\right)}-\frac{1}{x}\right) \\
&= \lim_{x\to 0}\frac{2+x^2+x^4/12+\cdots}{2x+x^3/3+\cdots}-\frac{1}{x} \\
&= \lim_{x\to 0}\frac{x\left(2+x^2+x^4/12+\cdots\right)-\left(2x+x^3/3+\cdots\right)}{x\left(2x+x^3/3+\cdots\right)} \\
&= \lim_{x\to 0}\frac{\left(2x+x^3+x^5/12+\cdots\right)-\left(2x+x^3/3+\cdots\right)}{x\left(2x+x^3/3+\cdots\right)} \\
&= \lim_{x\to 0}\frac{2/3x^3+\cdots}{2x^2+x^4/3+\cdots} \\
&= 0
\end{aligned}$$