This is a set of drills for practice of algebraic manipulation. Do not use a calculator except possibly for verification. (Try to keep numerical fractions together; this is a rough æsthetic guideline.)

1. Firstly, rules of algebraic manipulation
   (a) $a + b = b + a$
   (b) $ab = ba$
   (c) $a + (b + c) = (a + b) + c$
   (d) $a(bc) = (ab)c$
   (e) $a(b + c) = ab + ac$
   (f) $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
   (g) $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$
   (h) $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$
   (i) $$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\frac{d}{d} + \frac{b}{b}\frac{c}{d} = \frac{ad+bc}{bd}$$
   (j) also observe that $a \cdot b = 0 \leftrightarrow a = 0, b = 0,$ or $a = b = 0$.

2. Write as a single fraction
   (a) $4 + \frac{1}{2}$
   (b) $3 \times \frac{1}{7}$
   (c) $\frac{4}{8} \times \frac{3}{4}$
   (d) $\frac{4}{8} \div \frac{3}{4}$
   (e) $16 \times \frac{3}{2} - \frac{4}{16} \times \frac{2^4}{\sqrt{16}}$
   (f) $16 \times \frac{2}{x} + \frac{x^2+2x+1}{x(x+1)(x-1)}$
   (g) $1 - \frac{1}{x-1}$
   (h) $1 + \frac{1}{x-1}$
   (i) $\frac{x^3+3x^2+3x+1}{(x+1)^4} - \frac{3x}{(x+1)}$
   (j) $\frac{4x^3+2x^2+6x+3}{x(2x+1)} - \frac{2x^2+3}{x}$
   (k) $\frac{\frac{1}{x} \times \frac{2x+1}{3}}{\frac{3x+1}{\ln e^{3x}}} - \frac{2xe^x}{3 \ln e^x}$
   (l) $\frac{2x+3x^2}{2x} - \frac{3x^3}{2x+1}$
   (m) $\frac{2x(2+6x)-2(2x+3x^2)}{(2x)^2} - \frac{(2x+1)(9x^2)-(2)(x^2)}{(2x+1)^2}$
   (n) $\frac{2(3x+1)}{(3x+1)^2} - \frac{3(2x+1)}{(3x+1)^2}$

3. Now, rules of exponentiation:
   (a) $(xy)^n = x^n y^n$
   (b) $(x/y)^n = x^n/y^n$
   (c) $x^n x^m = x^{n+m}$
   (d) $x^n/x^m = x^{n-m}$
   (e) $(x^a)^b = x^{ab}$
   (f) $x^{a/b} = (x^a)^{1/b} = (x^{1/b})^a$
   (g) $x^{-a} = 1/x^a = (1/x)^a$
   (h) $x^{1/n} = \sqrt[n]{x}$

4. Simplify the following. Remember that the simplest form requires the fewest operations for evaluation of the form.
   (a) $x^3 x^4$
   (b) x^2/x^2
   (c) x^2/x^3
   (d) $(2^3 3^2)^5$
   (e) $(3^2 3^5)^3$
   (f) $(x^3 x^5)^7$
   (g) $(x/y)^{17}$
   (h) $5^{x-2} 25^2$
   (i) $x^{3-y} x^y$
   (j) $(x^{1/n})^{2n}$
   (k) $3x^2 + 2x^2$
   (l) $3x^2 + 2x^3$
   (m) $a^{\sqrt{2}+1} - a^{\sqrt{2}}$
   (n) $e^2(e^3 - 1/e^3)^2$
   (o) $e^{-6}(e^3 - 1/e^6)^2$
   (p) $\sqrt{e^2}(e^3 - 1/e^3)$
   (q) $e^{-2}(e^3 - 1/e^3)^{1/2}$

(r) $e^3 e^{-3}$

(s) $3^x(3^2-1)$

(t) $7^{-x}(7^x-1)$

(u) 10^3+10^4

(v) $10^3 10^4$

5. Rules of logarithmic manipulation (observe that we don't have one for a sum inside of a logarithm, but we have one for the sum of logarithms).

   (a) Definition

   $$\log_b x = y \longleftrightarrow b^y = x.$$

   (b) Product rule of logarithms

   $$\log_b x + \log_b y = \log_b(xy).$$

   (c) Quotient rule of logarithms

   $$\log_b x - \log_b y = \log_b \frac{x}{y}.$$

   (d) Power rule of logarithms

   $$\log_b(x^y) = y \log_b(x).$$

   (e) Rule for the change of base

   $$\log_N M = \frac{\log_b M}{\log_b N}.$$

   (You may choose any new base b)

6. Simplify the following

   (a) $\frac{\ln(\sqrt{1024^x})}{3} + \frac{x}{25}$

   (b) $\frac{\ln 2^x}{(x+1)} + \frac{\ln 4^{x/2}}{(x-1)}$

   (c) $\frac{\ln 2^{4x^3+2x^2+6x+3}}{(2x^2+3)} \div \frac{2}{e^x}$

   (d) $\frac{3e^x(2/x)}{(3e^x)^2} - \frac{(2\ln x)(3e^x)}{(3e^x)^2}$

7. Simplify the following

   (a) $\log_{10} 100$

   (b) $\ln 7 + \ln 10$ (Note that 'ln' is nothing more than '$\log_e$')

   (c) $\log_{10} 1000 - \log_{10} 1000$

   (d) $\log_2 16 - 4\log_2 256$

   (e) $2\ln x - 5\ln(4x)$

   (f) $\frac{\log_{10} 27}{\log_{10} 3}$

   (g) $\frac{\ln 64}{\ln 2} - \frac{\log_{12} 49}{\log_{12} 7}$

   (h) $(1/2)\log_6 36$

   (i) $(1/7)\log_3 e$

   (j) $\ln((1/7)\ln e^7)$

   (k) $5\ln(e^{1/5})$

   (l) $5\ln(5) + (1/5)\ln(1/5)$

   (m) $\frac{\log_a b}{\log_b a}$

   (n) $\ln 4 - x\ln 5$

   (o) $\log_3(7-w) + \log_3(7+w)$

   (p) $\frac{\log_{10}(x)+\log_{10}(y^2)}{\frac{\ln x}{\ln 10} - \frac{\ln y}{\frac{1}{2}\ln 10}}$

   (q) $\log_3(x^y)^{(1/y)} - \log_3 x$

   (r) $\ln e$

   (s) $\log_8 8^{493}$

   (t) $\ln e^{(2x^2-17)}$

   (u) $e^{\ln(23\sqrt{x}-24x^3)}$

   (v) $100^{\log_{10} 10^{(2x^3-17xy)}}$

   (w) $2^{\ln e^2} - 4^{(\ln e^{1/2})}$

   (x) $\frac{e^{\ln(x^2+2x+1)}}{e^{\ln((x+1)(x+2))}}$

8. Simplify these slightly more complicated expressions

   (a) $\ln 25^{\log_5 \log_3 3^5} - 2\ln 5$

   (b) $\frac{e^{\ln 25}}{\log_{10} 100000}$

   (c) $(2^{\ln e^{25}})^{3/(\ln e^{25} \log_{10} 1000)}$

   (d) $\sqrt[75]{(2^{3\ln e^{25}})}$

   (e) $\log_3 3^{(e^{-2})}(e^2-e^3) - \log_{10} 10 + \ln e$

   (f) $35^{\log_x x} - x^{(\log_x 7 + \log_x 5)}$

   (g) $\log_x x - \log_x 7 + \log_x 5$

   (h) $32^{4/5} + (-6)^0 - (\frac{1}{9})^{-1/2} + \sqrt[3]{27^2}$

   (i) $\log_{10}\frac{1}{2} + \log_{10}\frac{2}{3} + \log_{10}\frac{3}{4} + \cdots + \log_{10}\frac{9}{10}$

(j) $\frac{\sqrt{8^{10}+4^{10}}}{\sqrt{8^4+4^{11}}}$

(k) $2^{(\log_2 51+\log_2 31)}$

9. The purpose of the above was to allow you to become accustomed to simple manipulations, which are ultimately used in the solutions of equations. Solve for x in each of the following!

   (a) $x^{1/3} = 27^{2/3}$

   (b) $\log_8 x + \log_8(1/6) = 2/3$

   (c) $\log_5(\log_3 x) = 1$

   (d) $5^{2x} - 7 \cdot 5^x + 6 = 0$

   (e) $\log_3(x-6) + \log_3(x-14) = 2$

   (f) $9^x - 9^{x-1} = 72$

   (g) $\log_5(x+9) = 2\log_5(x-3)$

   (h) $(\log_{10} x)(\log_5 10) = 3$

   (i) $e^x - 3e^{-x} = 2$

   (j) $(\ln x)^2 = \ln x^2$

   (k) $10^{2x} = (2/5)\ln e^{25}$

   (l) $5^x - 5^{x-2} = 15000$

   (m) $2^x + 2^{-x} = 17/4$

10. Solve each of the following equations and remember that, whatever you do to one side of an equation, you must do to the other side! Some of these are really tricky; you might skip this section and return to it later!

    (a) given that $x = -3$ is a root of the equation $2x^3 + 7x^2 - 9 = 0$, determine the other roots

    (b) $5x^3 - 8x^2 - 2x + 3 = 0$

    (c) solve the system $3x^2+5xy-2y^2 = -15$ and $x + 2y = 3$

    (d) $|x+2| = 2|x-1|$

    (e) $(2x-1)(x+3) = 4x$

    (f) $\frac{x}{x-1} - \frac{1}{x} = \frac{3}{2}$

    (g) $6x^3 + 13x^2 = 2 - x$

    (h) $\frac{2x-1}{x-4} = 1$

    (i) $\frac{2-y}{y-2} = \frac{y}{2}$

    (j) here's an inequality: $x(x^2-9) \geq 0$

11. Solve each of the following equations and remember that, whatever you do to one side of an equation, you must do to the other side!

    (a) $5^x - 5^{x-2} = 15000$

    (b) $\ln e^x = 2^{\log_2 3+\log_2 5}$

    (c) $2 + \sqrt{2x+1} = x + 1$

    (d) $x = \frac{\log_{10} 32}{\log_{10} 2}$

    (e) $9^x - 9^{x-1} = 648$

    (f) solve the system $xy^2 = 10^8$ and $\frac{x^3}{y} = 10^{10}$

    (g) solve the system $y = \log_2(2x-6)$ and $y = \log_4 x$

    (h) try the inequality $e^{1-3x} < 1$

12. Again, for what real values of x are the following satisfied?

    (a) $3a^{2x} + 3a^{-2x} = 10$

    (b) $2^x + 4^x = 8^x$

    (c) $\log_x 2 = \log_{2x} 8$

    (d) $3^{x+4} = 7^{x-1}$

    (e) $\log_{10}(2+x) = -1$

    (f) $10^{3x} = 5$

    (g) $\log_{10}(x^2+2x+1) = 1$

    (h) $\ln(x^2+2x+10) = 1$

    (i) $10^{5-x^2} = 100$

    (j) $10^{1-x^2} = 100$

    (k) $\log_{10}(x-3) + \log_{10} x = 1$

    (l) $\log_{10}(3-x) + \log_{10} x = 1$

    (m) $\log_{10}[x(x-3)] = 1$

    (n) $\log_a x + \log_a(x+2) = 2$

Answers!

1. Ansers for section 2
   (a) $9/2$
   (b) $22/7$
   (c) $3/8$
   (d) $1/3$
   (e) 23
   (f) $\frac{32x-31}{x(x-1)}$
   (g) $\frac{x-2}{x-1}$
   (h) $x/(x-1)$
   (i) $\frac{1-3x}{1+x}$
   (j) 0
   (k) $\frac{6x+3-6xe^x-2e^x}{9x+3}$
   (l) $\frac{-6x^3+6x^2+7x+2}{4x+2}$
   (m) $\frac{-36x^3-2x^2+12x+3}{2(2x+1)^2}$
   (n) $-1/(3x+1)^2$
2. Answers for section 4
   (a) x^7
   (b) 1
   (c) x^{-1}
   (d) 1981355655168
   (e) 3^{21}
   (f) x^{56}
   (g) already simple
   (h) 5^{x+2}
   (i) x^3
   (j) x^2
   (k) $5x^2$
   (l) $x^2(3+2x)$
   (m) $a^{\sqrt{2}}(a-1)$
   (n) $(e^4-e^{-2})^2$
   (o) $(1-e^{-6})^2$
   (p) e^4-e^{-2}
   (q) $(e^{-1}-e^{-7})^{1/2}$
   (r) 1
   (s) $3^x(3^2-1)$
   (t) $1-7^{-x}$
   (u) 11000
   (v) 10000000
3. Answers for section 6
   (a) $\frac{125\ln 2+3}{75}x$
   (b) $\frac{2x^2\ln 2}{(x+1)(x-1)}$
   (c) $xe^x\ln 2$
   (d) $\frac{2(1-x\ln x)}{3xe^x}$
4. Answers for section 7
   (a) 2
   (b) $\ln 70$
   (c) 0
   (d) -28
   (e) $\ln(\frac{1}{1024}x^{-3})$
   (f) 3
   (g) 4
   (h) 3
   (i) $\frac{1}{7}\log_3 e$
   (j) 0
   (k) 1
   (l) $\frac{5}{9}\ln 5$
   (m) $(\log_b a)^{-2}$
   (n) $\ln(4-5^x)$
   (o) $\log_3(49-w^2)$
   (p) 1
   (q) 0
   (r) 1
   (s) 493
   (t) $2x^2-17$
   (u) $23\sqrt{x}-24x^3$
   (v) $10^{(4x^3-28xy)}$

(w) 2

(x) $(x+2)^{-1}$

5. Answers for section 8

(a) 0

(b) 5

(c) 2

(d) 2

(e) $1-e$

(f) 0

(g) $1+\log_x \frac{5}{7}$

(h) 23

(i) -1

(j) 16

(k) 1581

6. Answers for section 9

(a) $x = 3^6$

(b) $x = 6\sqrt[3]{2}$

(c) $x = 3^5$

(d) $x = 0$ or $\log_5 6$

(e) $x = 5$ or 15

(f) $x = 2$

(g) $x = 0$ or 7

(h) $x = 5^3$

(i) $x = \ln 3$

(j) $x = 1$ or e^2

(k) $x = 1/2$

(l) $x = 6$

(m) $x = \log_2 \frac{17}{8} \pm \frac{1}{2}\sqrt{285}$

7. Answers for section 10

(a) $x = 1, -3/2,$ or -3

(b) $x = 3/5$ or $\frac{5}{2}(1 \pm \sqrt{5})$

(c) $y = 2, x = -1$

(d) $x = 4$ or 0

(e) $x = -3/2$ or 1

(f) $x = -1$ or 2

(g) $x = -1/2$ or $-5/6 \pm \sqrt{286}/12$

(h) $x = -3$

(i) $y = \pm 2$

(j) $-3 \le x \le 0$ or $3 \le x$

8. Answers for section 11

(a) $x = 6$

(b) $x = 15$

(c) $x = 0$ or 4

(d) $x = 5$

(e) $x = \log_9(324 \pm \frac{1}{2}\sqrt{652})$

(f) $x = 10^4, y = 10^{-6}$

(g) $x = 4$ or $9/4$

(h) $x > 1/3$

9. Answers to section 12

(a) $x = \pm \log_a \sqrt{3}$

(b) $\log_2 \frac{1 \pm \sqrt{5}}{2}$

(c) $x = \pm\sqrt{2}$

(d) $x = \frac{\log_3 7 + 4}{\log_3 7 - 1}$

(e) $x = -19/20$

(f) $x = \frac{1}{3}\log_{10} 5$

(g) $x = -1$

(h) no real solution

(i) $x = \pm\sqrt{3}$

(j) no real solution

(k) $x = 5$ or -2

(l) no real solution

(m) $x = 5$ or -2

(n) $x = -1 \pm \sqrt{1 + 9^2/2}$