Prove that $\left|\ln\left(e+\frac{1}{n}\right)-1\right|$ is eventually less than $1/100$.

Proof:
We must show that we have a natural number N sufficiently large for which

$$\left|\ln\left(e+\frac{1}{n}\right)-1\right|<\frac{1}{100}$$

for all $n \geq N$. We shall consider the general case, that

$$\left|\ln\left(e+\frac{1}{n}\right)-1\right|<\epsilon$$

for all $n \geq N$. In other words that $\ln\left(e+\frac{1}{n}\right)$ is eventually arbitrarily close to 1 within an error ϵ. This implies the desired result with $\epsilon = 1/100$.

Firstly, we should examine the expression $\left|\ln\left(e+\frac{1}{n}\right)-1\right|<\epsilon$. Observe the following:

$$\begin{aligned}\left|\ln\left(e+\frac{1}{n}\right)-1\right| &= \left|\ln\left(e+\frac{1}{n}\right)-\ln(e)\right| \\ &= \left|\ln\left(\frac{e+\frac{1}{n}}{e}\right)\right| \\ &= \left|\ln\left(1+\frac{1}{en}\right)\right|\end{aligned}$$

and, because $n \geq 1$, $1+1/en>1$, so $\ln(1+1/en)>0$, thus

$$\left|\ln\left(e+\frac{1}{n}\right)-1\right|=\ln\left(1+\frac{1}{en}\right).$$

Now, if $\ln\left(1+\frac{1}{en}\right)<\epsilon$, $1+1/en<e^{\epsilon}$ or $1/en<e^{\epsilon}-1$. Additionally, because $\epsilon>0$, $e^{\epsilon}>1$, so $e^{\epsilon}-1>0$. Thus we have

$$\frac{1}{e(e^{\epsilon}-1)}<n.$$

$\mathbb{R}$ is an Archimedean field.[1] Consequently a natural number n does in fact satisfy $\frac{1}{e(e^{\epsilon}-1)}<n$, call it N.
Claim: for arbitrary $\epsilon>0$, with N as a defined above, whenever $n \geq N$, $\left|\ln(e+\frac{1}{n})-1\right|<\epsilon$. We demonstrate this.
Fix $\epsilon>0$. Let N be a natural number for which $\frac{1}{e(e^{\epsilon}-1)}<N$. Let n be an arbitrary natural larger

[1] $x, y \in \mathbb{R}$ with $x, y>0 \longrightarrow \exists n \in \mathbb{N}(nx>y)$. Geometrically, this entails that a line of arbitrary length (y) may be covered by a finite number (n) of line segments of a given length (x).

or equal to N. We have the following:

$$n \geq N > \frac{1}{e(e^{\epsilon}-1)}$$
$$e^{\epsilon}-1 > \frac{1}{ne}$$
$$e^{\epsilon} > \frac{1}{ne}+1$$

now, because ln is monotonic,

$$\ln(e^{\epsilon}) > \ln\left(\frac{1}{ne}+1\right) = \left|\ln\left(\frac{1}{ne}+1\right)\right|$$

so

$$\epsilon > \left|\ln\left(\frac{1}{ne}+1\right)\right|$$

but $\left|\ln\left(\frac{1}{ne}+1\right)\right| = \left|\ln\left(\frac{\frac{1}{n}}{e}+\frac{e}{e}\right)\right| = \left|\ln\left(\frac{\frac{1}{n}+e}{e}\right)\right|$

$$= \left|\ln\left(\frac{1}{n}+e\right) - \ln(e)\right| = \left|\ln\left(\frac{1}{n}+e\right)-1\right|$$

so

$$\epsilon > \left|\ln\left(\frac{1}{n}+e\right)-1\right|.$$

Finally, because $n \geq N$ was arbitrary, $\epsilon > \left|\ln\left(\frac{1}{n}+e\right)-1\right|$ must be true for all $n \geq N$, as desired.