The Frenet-Serret formulae relate the derivatives of the Frenet vectors to linear combinations of them. The Frenet frame is the triplet of Frenet vectors which form the orthonormal basis of vectors following the trajectory of a point along the parametrized path of its motion in $\mathbb{R}^3$. The equations are named after Jean Frédéric Frenet (1816 - 1900), who was a French mathematician, astronomer, and meteorologist; and Joseph Alfred Serret (1819 - 1885) who was a french mathematician, born in Paris and died in Versailles.

The Formulae

Specify C to be a space-curve in $\mathbb{R}^3$ which has a parametrization $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$ on $[\alpha, \beta]$. Suppose that $s = s(\nu)$ is the distance parameter along C (beginning at $t = \alpha$), then

$$s(\nu) = \int_\alpha^\nu \left\| \frac{d\mathbf{r}(t)}{dt} \right\| dt\,.$$

We now define the unit tangent along the curve to be

$$\widehat{\mathbf{T}} = \frac{d\mathbf{r}}{ds}\,.$$

From the first fundamental theorem of calculus we have

$$\frac{ds(\nu)}{d\nu} = \frac{d}{d\nu} \int_\alpha^\nu \left\| \frac{d\mathbf{r}(t)}{dt} \right\| dt = \left\| \frac{d\mathbf{r}(\nu)}{d\nu} \right\|\,.$$

Thus we have

$$\widehat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\nu}\frac{d\nu}{ds} = \frac{d\mathbf{r}(\nu)}{d\nu} \Bigg/ \left\| \frac{d\mathbf{r}(\nu)}{d\nu} \right\|\,.$$

The normal vector is defined to be

$$\mathbf{N} = \frac{d\widehat{\mathbf{T}}}{ds},$$

and thus the unit normal is

$$\widehat{\mathbf{N}} = \frac{d\widehat{\mathbf{T}}}{ds} \Bigg/ \left\| \frac{d\widehat{\mathbf{T}}}{ds} \right\|.$$

That may be rearranged to give

$$\frac{d\widehat{\mathbf{T}}}{ds} = \left\| \frac{d\widehat{\mathbf{T}}}{ds} \right\| \widehat{\mathbf{N}},$$

and when we define the curvature, κ, to be $\kappa = \frac{d\widehat{\mathbf{T}}}{ds}$ (the radius of curvature ρ is $\rho = 1/\kappa$), the equation becomes

$$\frac{d\widehat{\mathbf{T}}}{ds} = \kappa \widehat{\mathbf{N}}\,.$$

The plane which contains $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ is called the osculating plane, and the circle with tangent $\widehat{\mathbf{T}}$ and radius equal to the radius of curvature ρ is called the osculating circle.

The binormal is defined to be

$$\widehat{\mathbf{B}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}}\,,$$

from which we also have

$$\widehat{\mathbf{N}} = \widehat{\mathbf{B}} \times \widehat{\mathbf{T}},$$

and

$$\widehat{\mathbf{T}} = \widehat{\mathbf{N}} \times \widehat{\mathbf{B}}.$$

Differentiation of these expressions yields the Frenet-Serre formulae. The derivative of $\widehat{\mathbf{B}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}}$ yields $\widehat{\mathbf{B}}' = \widehat{\mathbf{B}}' \times \widehat{\mathbf{N}} + \widehat{\mathbf{T}} \times \widehat{\mathbf{N}}'$, but $\widehat{\mathbf{T}}'$ is parallel to $\widehat{\mathbf{N}}$ and thus the first term vanishes. Thus the binormal must be orthogonal to $\widehat{\mathbf{T}}$, but also to $\widehat{\mathbf{N}}'$, and, consequently,

$$\widehat{\mathbf{B}}' = \tau \widehat{\mathbf{N}}$$

in which τ, called the torsion, is simply $\tau = \left\| \frac{d\widehat{\mathbf{B}}}{ds} \right\|$. The derivative of $\frac{d\widehat{\mathbf{N}}}{ds}$ will yield a linear combination of $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{B}}$. We differentiate to get $\frac{d\widehat{\mathbf{N}}}{ds} = \widehat{\mathbf{B}} \times \widehat{\mathbf{T}}' + \widehat{\mathbf{B}}' \times \widehat{\mathbf{T}}$. Upon substitution, we get $\widehat{\mathbf{N}}' = \kappa \widehat{\mathbf{B}} \times \widehat{\mathbf{N}} + \tau \widehat{\mathbf{N}} \times \widehat{\mathbf{T}}$ or $\widehat{\mathbf{N}}' = -\kappa \widehat{\mathbf{T}} - \tau \widehat{\mathbf{B}}$.
The Frenet-Serre formulae are therefore

$$\frac{d\widehat{\mathbf{T}}}{ds} = \kappa \widehat{\mathbf{N}}\,,$$

$$\frac{d\widehat{\mathbf{B}}}{ds} = -\kappa \widehat{\mathbf{T}} - \tau \widehat{\mathbf{B}}\,, \ \textit{and}$$

$$\frac{d\widehat{\mathbf{B}}}{ds} = \tau \widehat{\mathbf{N}}\,.$$

In matrix form, they are

$$\begin{pmatrix} \widehat{\mathbf{T}}' \\ \widehat{\mathbf{N}}' \\ \widehat{\mathbf{B}}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}} \\ \widehat{\mathbf{N}} \\ \widehat{\mathbf{B}} \end{pmatrix}$$

The unit tangent, $\widehat{\mathbf{T}}$, unit normal, $\widehat{\mathbf{N}}$, and binormal unit vector, $\widehat{\mathbf{B}}$, are collectively called the Frenet vectors. Together, they form an orthonormal basis for $\mathbb{R}^3$, a reference frame which follows the trajectory of a point. The reference frame is called a Frenet frame, and it is neither static nor inertial because the Frenet frame moves tangentially to the (non-straight) curve, so it is constantly accelerating. The curve can be assumed to be parametrically dependent on time, i.e. the Frenet frame can be visualised kinematically.

Exercise

For the circular helix, $\mathbf{x}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$ with $a, b > 0$ and $0 \le t < \infty$, derive the following:

$$s(t) = \sqrt{a^2 + b^2}\, t\,,$$

$$\widehat{\mathbf{T}}(t) = c(-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k})\,,$$

$$\widehat{\mathbf{N}}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j})\,,$$

$$\widehat{\mathbf{B}}(t) = c(b\sin t\mathbf{i} - b\cos t\mathbf{j} + a\mathbf{k})\,,$$

$$\kappa(t) = a/(a^2 + b^2)\,,$$

$$\rho(t) = (a^2 + b^2)/a\,, \text{ and}$$

$$\tau(t) = -b/(a^2 + b^2)\,,$$

in which $c = 1/\sqrt{a^2 + b^2}$.