Symmetry

Graphs of functions might exhibit symmetry. Two kinds of symmetry will be important in the development which follows. Study the plots of powers of x figures 1 and 2 below. What kind of symmetry do you observe? How might you characterize the symmetry in a succinct mathematical expression? What other familiar functions exhibit the same kinds of symmetry?

Reflective symmetry through the origin — Odd Symmetry

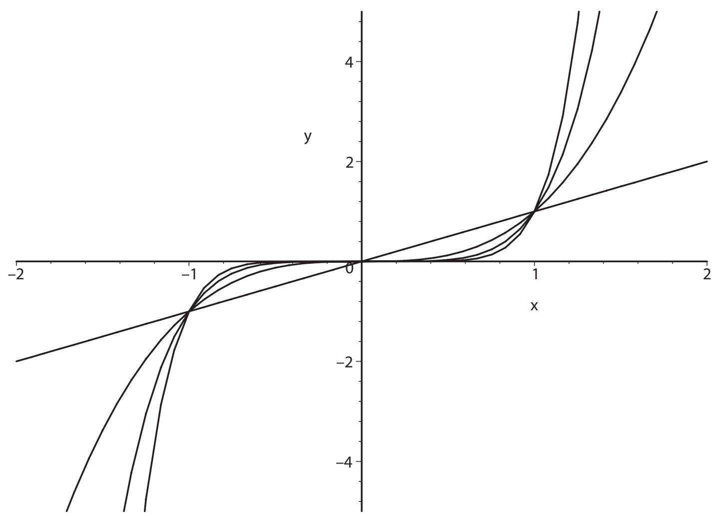

Figure 1: Plot of odd powers of x.

In figure 1, we notice that all of the odd powers of x shown have reflective symmetry through a point, in this case, through the origin. It should be evident that every odd power exhibits this kind of symmetry. Other functions exhibit this kind of symmetry, and when they do, they are said to be "odd functions" because their symmetry is analogous to the *odd* powers of x. We may characterize this symmetry algebraically by stating that $f(x)$ is odd if and only if $f(-x) = -f(x)$ for all $x \in \mathcal{D}f$. Why does this make sense? If we locate points on the x-axis symmetrically about the origin, we will notice that the magnitudes of the values of f are the same their signs are opposite.

Reflective symmetry through the y-axis — Even Symmetry

In figure 2, we notice that all of the even powers of x shown have reflective symmetry through an axis, in this case, through y-axis. It should be evident that every even power exhibits this kind of symmetry. Many other functions exhibit this kind of symmetry, and when they do, they are said to be "even functions" because their symmetry is analogous to that of the *even* powers of x. We may characterize this symmetry algebraically by stating that $f(x)$ is even if and only if $f(-x) = f(x)$ for all $x \in \mathcal{D}f$. Why does this make sense? If we locate points on the x-axis symmetrically about the origin, we will notice that the magnitudes of the values of f are the same and their signs are the same as well.

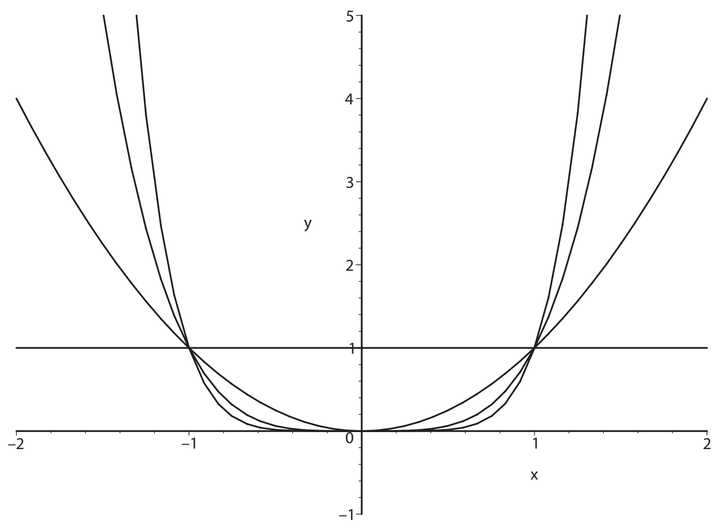

Figure 2: Plot of even powers of x.

Sine and Cosine

Sine is an odd function, mathematically, $\sin(-\theta) = -\sin\theta$ for all $\theta \in \mathbb{R}$. Cosine is an even function, mathematically, $\cos(-\theta) = \cos\theta$ for all $\theta \in \mathbb{R}$. These facts should be evident from the fact that the Maclaurin series for cosine contains only even powers of x and the Maclaurin series for sine contains only odd powers of x. A quick plot of the graphs of sine and cosine should also confirm this symmetry.

Trigonometric Identities

Euler's Formula

If we accept the definition of the exponential function, defined for any complex number $z \in \mathbb{C}$, to be

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1}$$

then, for purely imaginary $z = ix$, upon substitution of this into (1), we have

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix + \frac{i^2x^2}{2} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots \end{aligned}$$

but $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = (-1)(-1) = 1$, $i^5 = i^4 i = i$, etc., so we have

$$e^{ix} = 1 + ix - \frac{x^2}{2} + -i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

and now, upon grouping of the real and imaginary parts, we obtain

$$e^{ix} = (1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

but the real part is simply the Maclaruin series for cosine and the imaginary part is simply the Maclaurin series for sine. In other words, because

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{and} \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,,$$

we must have

$$e^{ix} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos x + i \sin x\,.$$

Thus we obtain Euler's Formula[1] (in its familiar form with x replaced with θ),

$$\boxed{e^{i\theta} = \cos\theta + i\,\sin\theta} \tag{2}$$

A brief digression on a matter of logical presentation

The astute reader, one who has looked ahead to see the presentation of derivatives follows in a later section, might wonder on what grounds may we accept the Maclaurin series for sine and cosine. This is because the typical presentation, the presentation one often encounters in elementary calculus, is that the formulae for the series representations of sine and cosine are consequences of the respective derivatives applied repeatedly and evaluated at zero, $\sum_{n=0}^{\infty} f^n(0)/n!\, x^n$. However, if we accept (1) and we understand $e^{i\theta}$, in the case in which $\theta \in \mathbb{R}$, to be a complex number of unit length at polar angle θ (positive angle is measured counter clockwise from the positive real axis), then if we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

we are able to recover all of the properties of sine and cosine, symmetries, periodicity, and, without differentiation, the Maclaurin series for sine and cosine. This makes perfect sense because, under this set of definitions, $\cos\theta$ is simply the real part of $e^{i\theta}$, the projection of the ray at θ onto the real axis, and $\sin\theta$ is simply the imaginary part of $e^{i\theta}$, the projection of the ray at θ onto the imaginary axis. $\cos z$ is the real part of e^{iz} and $\sin z$ is the imaginary part of e^{iz} due to the fact that $\Re(z) = (z + \bar{z})/2$ and $\Im(z) = (z - \bar{z})/2i$.

[1]Leonhard Paul Euler (1707 – 1783) was born in Switzerland. He aggressively developed a tremendous amount of mathematics and applied it to the study of physical phenomena. Euler contributed to mathematical areas as diverse as calculus, number theory, and graph theory. He is renowned for his work in mechanics. In fact, in terms of mathematical content, Mechanical Engineering is essentially a course in things Euler knew or things for which he layed the groundwork. I find this really humbling and yet inspiring. Euler is considered to be the preeminent mathematician of the 18th century and one of the greatest of all time. He is certainly amongst the most prolific mathematicians; his collected works number 25,000 pages, part of which was produced while he was effectively blind. Laplace suggests 'Read Euler, read Euler, he is the master of us all'. I suggest that you simply know Euler's formula and some of its consequences presented herein.

The angle-sum identities

The angle sum identities are a consequence of Euler's Formula, (2), and all of the subsequent identities are an immediate consequence of these!

A brief digression on $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$

From (1), we have [2]

$$\begin{aligned} e^{z_1}e^{z_2} &= \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \frac{z_1^r}{r!}\frac{z_2^{n-r}}{(n-r)!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \frac{1}{r!(n-r)!} z_1^r z_2^{n-r}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} z_1^r z_2^{n-r}\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{r=0}^{n} \binom{n}{r} z_1^r z_2^{n-r}\right) \end{aligned}$$

You must recognize the binomial theorem which, extended to $a, b \in \mathbb{C}$, is

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} e^{z_1}e^{z_2} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{r=0}^{n} \binom{n}{r} z_1^r z_2^{n-r}\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(z_1+z_2)^n = \sum_{n=0}^{\infty}\frac{(z_1+z_2)^n}{n!} \\ &= e^{z_1+z_2} \text{ by (1)} \end{aligned}$$

So we have proven that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. When we specialize this to the purely imaginary case by the following substitutions, that $z_1 = i\alpha$ and $z_2 = i\beta$ ($\alpha, \beta \in \mathbb{R}$), we obtain

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}\,. \tag{3}$$

[2] See Walter Rudin's *Principles of Mathematical Analysis*, third edition published by McGraw-Hill, Inc. in 1976 on page 74 for a discussion of the fact that the product of two convergent series converges to the Cauchy product, the value to which we expect the product to converge, as long as at least one of the series converges absolutely. Now, both of the series in the first line converge absolutely for all complex numbers, so this condition is clearly satisfied. Thus this proof is rigorous.

The Identities

Expansion of each side of (3) is outrageously fruitful!

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \text{ yields} \\ \cos(\alpha+\beta) + i\,\sin(\alpha+\beta) &= (\cos\alpha + i\,\sin\alpha)\,(\cos\beta + i\,\sin\beta) \\ &= \cos\alpha\cos\beta + \cos\alpha(i\sin\beta) + (i\sin\alpha)\cos\beta + (i\sin\alpha)(i\sin\beta) \\ &= \cos\alpha\cos\beta + i^2\sin\alpha\sin\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta \quad \text{and finally} \\ \cos(\alpha+\beta) + i\,\sin(\alpha+\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \end{aligned}$$

When we equate the real and imaginary parts on the left and right hand sides (because two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal), we obtain the angle-sum identities for cosine

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{4}$$

and for sine

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \tag{5}$$

The angle-difference identities

From (4) and (5), we systematically develop the remaining identities, all of which are gathered at the end of this section on identities. We start by exploiting symmetry properties of sine and cosine upon replacement of β with $-\beta$. We have, upon substitution of $-\beta$ for β in (4),

$$\begin{aligned} \cos(\alpha+(-\beta)) &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta) \text{ because cos is even and sin is odd} \end{aligned}$$

Thus we have the angle difference identity for cosine,

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta} \tag{6}$$

Now, upon substitution of $-\beta$ for β in (5), we obtain

$$\begin{aligned} \sin(\alpha+(-\beta)) &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta) \text{ because cos is even and sin is odd} \end{aligned}$$

Thus we have the angle difference identity for sine,

$$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \tag{7}$$

Angle-sum and angle-difference identities for $\tan$ and $\cot$

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
&= \frac{\cos\alpha\cos\beta\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right)}{\cos\alpha\cos\beta\left(1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)} = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}}
\end{aligned}$$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}} \tag{8}$$

Now, set β to $-\beta$ as we have done before to go from the angle sum to the angle difference. Before we do this however, we must think about the symmetry properties of tan.

Proposition 1. *The quotient and product of an even and an odd function is odd.*

Proof. Suppose f is even and g is odd. Consider $(fg)(-x)$.

$$\begin{aligned}
(fg)(-x) &= f(-x)g(-x) && \text{by definition of the product} \\
&= f(x)(-g(x)) && \text{because } f \text{ is even and } g \text{ is odd} \\
&= -f(x)g(x) \\
&= -(fg)(x) && \text{again, by definition, and therefore} \\
(fg)(-x) &= -(fg)(x)
\end{aligned}$$

x was arbitrary, so this final equality holds for all x, so the product, in any order, must be odd.
Now we apply a similar argument for the quotient of an even function by an odd function, assuming $g(\pm x) \neq 0$ (a fact which must be true on the domain of the quotient of f by g),

$$\begin{aligned}
\left(\frac{f}{g}\right)(-x) &= \frac{f(-x)}{g(-x)} && \text{by definition of the quotient} \\
&= \frac{f(x)}{-g(x)} && \text{because } f \text{ is even and } g \text{ is odd} \\
&= -\frac{f(x)}{g(x)} \\
&= -\left(\frac{f}{g}\right)(x) && \text{again, by definition, and therefore} \\
\left(\frac{f}{g}\right)(-x) &= -\left(\frac{f}{g}\right)(x)
\end{aligned}$$

x was arbitrary, so this final equality holds for all x, so the quotient must be odd. Similarly, if we swap f and g, assuming $f(\pm x) \neq 0$, we obtain the result for the quotient of an odd by an even. We have proven what we had set out to prove. □

So, because tan is the quotient of sin and cos, an odd function and an even function, tan must be odd. Now we may proceed with our demonstration

$$\begin{aligned}
\tan(\alpha + (-\beta)) &= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} \\
\tan(\alpha - \beta) &= \frac{\tan\alpha + (-\tan\beta)}{1 - \tan\alpha(-\tan\beta)} \qquad \text{because tan is odd}
\end{aligned}$$

$$\boxed{\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}} \tag{9}$$

Now for cot, in some sense we seek the reciprocal of the result for tangent, so instead of factoring $\cos\alpha\cos\beta$ from the numerator and denominator, we factor $\sin\alpha\sin\beta$,

$$\begin{aligned}
\cot(\alpha + \beta) &= \frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)} \\
&= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} \\
&= \frac{\sin\alpha\sin\beta\left(\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - 1\right)}{\sin\alpha\sin\beta\left(\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}\right)} = \frac{\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta}{\sin\beta} - 1}{\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}}
\end{aligned}$$

$$\boxed{\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}} \tag{10}$$

now substitute $-\beta$ for β

$$\begin{aligned}
\cot(\alpha + (-\beta)) &= \frac{\cot\alpha\cot(-\beta) - 1}{\cot(-\beta) + \cot\alpha} \\
\cot(\alpha - \beta) &= \frac{\cot\alpha(-\cot\beta) - 1}{(-\cot\beta) + \cot\alpha} \qquad \text{because cot is odd}
\end{aligned}$$

$$\boxed{\cot(\alpha - \beta) = \frac{-\cot\alpha\cot\beta - 1}{-\cot\beta + \cot\alpha}} \tag{11}$$

The angle-difference identities for tangent and cotangent might also have been obtained by manipulation of ratios of the angle-difference identities for sine and cosine, (7) and (6).

Pythagorean Identities

We now continue systematic application of identities (4), (5), (6), and (7). Firstly, when we set β to α (and this is possible because these are identities, in other words, they are equations valid for all substitutions of the variables within the domains of the functions which comprise the expressions) and subtract, in other words, set β to α in (7), we obtain

$$\begin{aligned}
\sin(\alpha - \alpha) &= \sin\alpha\cos\alpha - \cos\alpha\sin\alpha \\
\sin(0) &= 0
\end{aligned}$$

and that is true but not particularly interesting, but at least it is consistent with our definition of the sine function and furnishes confirmation of the value of $\sin 0$. Now, when we set β to α in (6), we obtain

$$\begin{aligned}\cos(\alpha - \alpha) &= \cos\alpha\cos\alpha + \sin\alpha\sin\alpha \\ \cos(0) &= \cos^2\alpha + \sin^2\alpha\end{aligned}$$

and because cosine of zero is 1, we obtain the Pythagorean identity,

$$\boxed{\cos^2\alpha + \sin^2\alpha = 1} \tag{12}$$

Now, take (12) and either divide by $\cos^2\alpha$ or divide by $\sin^2\alpha$. We have

$$\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1 \quad \text{from (12)} \\ \frac{\cos^2\alpha}{\cos^2\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha} &= \frac{1}{\cos^2\alpha}\end{aligned}$$

$$\boxed{1 + \tan^2\alpha = \sec^2\alpha} \tag{13}$$

or

$$\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1 \quad \text{from (12)} \\ \frac{\cos^2\alpha}{\sin^2\alpha} + \frac{\sin^2\alpha}{\sin^2\alpha} &= \frac{1}{\sin^2\alpha}\end{aligned}$$

$$\boxed{\cot^2\alpha + 1 = \csc^2\alpha} \tag{14}$$

Double Angle Identities

Two different approaches to double and triple angle identities follow in this section and the next. Firstly we apply the angle sum identities, (4) and (5), and we subsequently employ De Moivre's Formula, another exciting result from the theory of complex numbers.

Double Angle Identities via Angle-Sum Identities

Set α and β to θ in the angle-sum formula for sine, formula (5), and obtain

$$\begin{aligned}\sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \quad \text{by (5)} \\ &= \sin\theta\cos\theta + \sin\theta\cos\theta\end{aligned}$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \tag{15}$$

the double angle formula for sine. Similarly set α and β to θ in the angle-sum formula for cosine, formula (4), and obtain

$$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \quad \text{by (4)}$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta} \tag{16}$$

the first double angle formula for cosine. To obtain two other double angle formulae for cosine, recall the Pythagorean identity, that $\sin^2\theta + \cos^2\theta = 1$, and rearrange to obtain either

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \quad \text{or} \\ \cos^2\theta &= 1 - \sin^2\theta \end{aligned}$$

and substitute these successively into (16).

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \quad \text{by (16)} \\ &= \cos^2\theta - (1 - \cos^2\theta) \quad \text{because } \sin^2\theta = 1 - \cos^2\theta \end{aligned}$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1} \tag{17}$$

and similarly

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \quad \text{by (16)} \\ &= (1 - \sin^2\theta) - \sin^2\theta \quad \text{because } \cos^2\theta = 1 - \sin^2\theta \end{aligned}$$

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta} \tag{18}$$

Now we develop an identity for the cosine function which involves the tangent function exclusively.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \quad \text{by (16)} \\ &= \cos^2\theta \left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right) \\ &= \frac{1 - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} \end{aligned}$$

but because

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \quad \text{and} \quad \frac{1}{\cos\theta} = \sec\theta$$

we have

$$\cos 2\theta = \frac{1 - \tan^2\theta}{\sec^2\theta}$$

and, finally, by the Pythagorean identity involving tangent and secant, (13), we obtain

$$\boxed{\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}} \tag{19}$$

We similarly develop an identity for the sine function which involves the tangent function exclusively.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \quad \text{by (15)}\\ &= 2\cos\theta\sin\theta = 2\cos\theta\left(\frac{\cos\theta}{\cos\theta}\right)\sin\theta\\ &= 2\cos^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= 2\frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\\ &= 2\frac{\tan\theta}{\sec^2\theta}\end{aligned}$$

$$\boxed{\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}} \tag{20}$$

Now for tangent and cotangent functions in two ways. We begin with tan.

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}\\ &= \frac{\cos^2\theta\left(2\frac{\sin\theta}{\cos\theta}\right)}{\cos^2\theta\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)}\end{aligned}$$

$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}} \tag{21}$$

or, alternatively, by the angle-sum identity for tangent,

$$\begin{aligned}\tan 2\theta = \tan(\theta+\theta) &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

or, yet another way, from the formulae for sine and cosine which involve tan exclusively,

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{2\tan\theta}{1+\tan^2\theta}}{\frac{1-\tan^2\theta}{1+\tan^2\theta}} = \frac{2\tan\theta}{1-\tan^2\theta}$$

Now for cotangent,

$$\begin{aligned}\cot 2\theta &= \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta}\\ &= \frac{\sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}-1\right)}{\sin^2\theta\left(2\frac{\cos\theta}{\sin\theta}\right)}\end{aligned}$$

$$\boxed{\cot 2\theta = \frac{\cot^2\theta-1}{2\cot\theta}} \tag{22}$$

or, alternatively, by the angle-sum identity for cotangent,

$$\cot 2\theta = \cot(\theta+\theta) = \frac{\cot\theta\cot\theta - 1}{\cot\theta + \cot\theta}$$
$$= \frac{\cot^2\theta - 1}{2\cot\theta}$$

Double Angle Identities via De Moivre's Formula

Proposition 2 (De Moivre's Formula[3]). *For $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$,*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{23}$$

Proof. This result is easily proven by induction on n. When $n=0$, the formula follows trivially, when $n=1$, again, the formula is trivially true. Thus we have addressed the truth of the base case, which we take to be $n=1$. Now suppose $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ for some $k \geq 1$, and try to show $(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta)$. We begin with the left hand side of this expression

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \quad \text{by hypothesis}\\
&= \cos k\theta\cos\theta + \cos k\theta(i\sin\theta) + i\sin k\theta\cos\theta + i\sin k\theta(i\sin\theta) \quad \text{upon expansion}\\
&= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \quad \text{because } i^2 = -1\\
&= \cos(k\theta+\theta) + i\sin(k\theta+\theta) \quad \text{by the angle-sum identities, (4) and (5)}
\end{aligned}$$

Thus, by induction, we have proven what we had set out to prove. □

An alternative proof is much more exciting, it demonstrates the truth of the formula for all real n, it is due to Euler, who demonstrated it in 1749, and, guess what, it employs Euler's formula, check it:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n\\
&= e^{in\theta}\\
&= \cos n\theta + i\sin n\theta
\end{aligned}$$

De Moivre's formula furnishes a way of expanding integral powers of complex numbers in polar form. We exploit its algebraic facility here and later when we prove the expansion identities. To develop the double angle formulae for sine and cosine essentially simultaneously, we simply expand both sides of (23) with $n=2$.

$$\begin{aligned}
\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \quad \text{by (23)}\\
&= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta\\
\cos 2\theta + i\sin 2\theta &= \cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta
\end{aligned}$$

and when we equate the real and imaginary parts, we obtain the double angle formulae (15) and (16).

[3]Abraham de Moivre (1667 – 1754) was a French mathematician who eventually lived in England. He is famous for this formula, and for his contributions to porbability theory. De Moivre made his living as a private tutor in mathematics. He was friends of Isaac Newton's, Edmund Halley's, and James Stirling's. In 1697, he was elected a Fellow of the Royal Society, but, despite his success, he was unable to secure an appointment to a university Chair of Mathematics which would have released him from his dependence on the burdensome tutoring which consumed his time.

Half Angle Identities

The results of this section are an immediate consequence of the double angle formulae we have developed. Take the second and third double angle identities for cosine, (17) and (18), and replace θ with $\theta/2$.

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \\
2\cos^2\theta &= 1 + \cos 2\theta \\
\cos^2\theta &= \frac{1+\cos 2\theta}{2}
\end{aligned}$$

and set θ to $\theta/2$ and obtain

$$\boxed{\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}} \tag{24}$$

For sine,

$$\begin{aligned}
\cos 2\theta &= 1 - 2\sin^2\theta \\
2\sin^2\theta &= 1 - \cos 2\theta \\
\sin^2\theta &= \frac{1-\cos 2\theta}{2}
\end{aligned}$$

and set θ to $\theta/2$ and obtain

$$\boxed{\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}} \tag{25}$$

For tangent, we need one extra result:

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \quad \text{by (15)} \\
\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad \text{after } \theta \text{ is replaced with } \theta/2 \\
\sin\frac{\theta}{2} &= \frac{\sin\theta}{2\cos\frac{\theta}{2}}
\end{aligned}$$

We are now ready for tangent:

$$\begin{aligned}
\tan\frac{\theta}{2} &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
&= \frac{\frac{\sin\theta}{2\cos\frac{\theta}{2}}}{\cos\frac{\theta}{2}} \quad \text{from the result above} \\
&= \frac{\sin\theta}{2\cos^2\frac{\theta}{2}} \\
&= \frac{\sin\theta}{2\frac{1+\cos\theta}{2}} \quad \text{by (24)}
\end{aligned}$$

$$\boxed{\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}} \tag{26}$$

We may easily obtain an alternate form:

$$\begin{aligned}\tan\frac{\theta}{2} &= \frac{\sin\theta}{1+\cos\theta}\\ &= \frac{\sin\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta}\\ &= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}\\ &= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}\end{aligned}$$

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}} \tag{27}$$

For cotangent, because

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}\,,$$

we take reciprocals and obtain

$$\boxed{\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}} \tag{28}$$

The Triple Angle Identities and the Expansion Identities

The development which immediately follows is essentially identical to the development of the double angle formulae, but later in this section, we develop a beautiful generalization, the expansion identities. For sine we have

$$\begin{aligned}\sin 3\theta = \sin(2\theta+\theta) &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos\theta\cos\theta + (1-2\sin^2\theta)\sin\theta\\ &= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta = 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta\\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta\end{aligned}$$

$$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta} \tag{29}$$

For cosine we have

$$\begin{aligned}\cos 3\theta = \cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\end{aligned}$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta} \tag{30}$$

Now, exactly as we have done previously with the double angle identities, we may develop the triple angle identities simultaneously by applying De Moivre's formula with $n = 3$.

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) \\
&= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) + i(3(1-\sin^2\theta)\sin\theta - \sin^3\theta) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta + i(3\sin\theta - 3\sin^3\theta - \sin^3\theta) \\
&= 4\cos^3\theta - 3\cos\theta + i(3\sin\theta - 4\sin^3\theta)
\end{aligned}$$

and upon equating of the real and imaginary parts on the left and right hand sides, we obtain exactly what we expect,

$$\begin{aligned}
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \quad \text{the real parts} \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta \quad \text{the imaginary parts}
\end{aligned}$$

We develop a triple angle formula for tangent, but not directly, I encourage you to try a direct approach as an exercise, but I in fact develop a much more sophisticated formula of which the triple angle formula for tangent is a special case. We proceed by discovering general expressions for sines and cosines of any integer multiple of an angle. These are the expansion identities. For the expansion identities for sine and cosine, we simply take De Moivre's formula, for general $n \in \mathbb{N}$, and apply the binomial theorem to obtain an expansion. Check it:

$$\begin{aligned}
\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n = (i\sin\theta + \cos\theta)^n \\
&= \sum_{r=0}^{n}\binom{n}{r}(i\sin\theta)^r(\cos\theta)^{n-r} = \sum_{r=0}^{n}\binom{n}{r}i^r\sin^r\theta\cos^{n-r}\theta \\
&= \sum_{\substack{r=0 \\ r \text{ is even}}}^{n}\binom{n}{r}i^r\sin^r\theta\cos^{n-r}\theta + \sum_{\substack{r=0 \\ r \text{ is odd}}}^{n}\binom{n}{r}i^r\sin^r\theta\cos^{n-r}\theta \\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}i^{2k}\sin^{2k}\theta\cos^{n-2k}\theta + \sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2l+1}i^{2l+1}\sin^{2l+1}\theta\cos^{n-(2l+1)}\theta \\
&= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(i^2)^k\sin^{2k}\theta\cos^{n-2k}\theta + \sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2l+1}i(i^2)^l\sin^{2l+1}\theta\cos^{n-2l-1}\theta \\
\cos n\theta + i\sin n\theta &= \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}(-1)^k\sin^{2k}\theta\cos^{n-2k}\theta + i\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2l+1}(-1)^l\sin^{2l+1}\theta\cos^{n-2l-1}\theta
\end{aligned}$$

and equating the real and imaginary parts on the left and right hand sides of the equation yields

$$\boxed{\cos n\theta = \sum_{k=0}^{\infty}(-1)^k \binom{n}{2k} \cos^{n-2k}\theta \, \sin^{2k}\theta} \tag{31}$$

$$\boxed{\sin n\theta = \sum_{l=0}^{\infty}(-1)^l \binom{n}{2l+1} \cos^{n-2l-1}\theta \, \sin^{2l+1}\theta} \tag{32}$$

The infinite upper limits are justified because all of the terms past a certain point, $\left\lfloor \frac{n}{2} \right\rfloor$ in the case of cosine and $\left\lfloor \frac{n-1}{2} \right\rfloor$ in the case of sine, will be zero anyway, this is a consequence of the definition of the binomial coefficient, specifically

$$\binom{a}{b} = 0 \quad \text{when } b > a\,.$$

Now, the expansion identity for tangent is obtained by dividing these, obviously, because tan is sin over cos. Duh![4]

$$\begin{aligned}
\tan n\theta &= \frac{\sin n\theta}{\cos n\theta} \\
&= \frac{\sum_{l=0}^{\infty}(-1)^l \binom{n}{2l+1} \cos^{n-2l-1}\theta \, \sin^{2l+1}\theta}{\sum_{k=0}^{\infty}(-1)^k \binom{n}{2k} \cos^{n-2k}\theta \, \sin^{2k}\theta}
\end{aligned}$$

we may factor $\cos^n\theta$ from each sum to obtain

$$\begin{aligned}
\tan n\theta &= \frac{\cos^n\theta \sum_{l=0}^{\infty}(-1)^l \binom{n}{2l+1} \cos^{-2l-1}\theta \, \sin^{2l+1}\theta}{\cos^n\theta \sum_{k=0}^{\infty}(-1)^k \binom{n}{2k} \cos^{-2k}\theta \, \sin^{2k}\theta} \\
&= \frac{\sum_{l=0}^{\infty}(-1)^l \binom{n}{2l+1} \frac{\sin^{2l+1}\theta}{\cos^{2l+1}\theta}}{\sum_{k=0}^{\infty}(-1)^k \binom{n}{2k} \frac{\sin^{2k}\theta}{\cos^{2k}\theta}}
\end{aligned}$$

$$\boxed{\tan n\theta = \frac{\sum_{l=0}^{\infty}(-1)^l \binom{n}{2l+1} \tan^{2l+1}\theta}{\sum_{k=0}^{\infty}(-1)^k \binom{n}{2k} \tan^{2k}\theta}} \tag{33}$$

Now, when we set $n = 3$ in (33), we obtain the triple angle identity for tangent. We have

$$\tan 3\theta = \frac{\sum_{l=0}^{\infty}(-1)^l \binom{3}{2l+1} \tan^{2l+1}\theta}{\sum_{k=0}^{\infty}(-1)^k \binom{3}{2k} \tan^{2k}\theta} = \frac{\binom{3}{1}\tan\theta + (-1)^1\binom{3}{3}\tan^3\theta}{\binom{3}{0} + (-1)^1\binom{3}{2}\tan^2\theta}$$

$$\boxed{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}} \tag{34}$$

[4]So I'm really condescending; I am justified because we've employed the ratio identity, the fact that tangent is sine over cosine many many times already. Furthermore, I haven't proven it in the current version of the document, maybe because it is so obvious. What? I must be joking, this is exactly the kind of sentiment I want to avoid, it is the exact bullshit which you encounter all of the time, so I'll provide a proper answer, here is the dealio: A, O, and H are the three sides of a right triangle, the *a*djacent, *o*pposite, and *h*ypotenuse respectively. (The hypotenuse is the long side opposite to the right angle, the adjacent side is one side together with the hypotenuse which forms the angle θ, and the opposite side is the one opposite to θ which closes the angle to form two new angles, thus completing the *tri*angle. We have definitions for a right triangle: $\sin\theta = O/H$, $\cos\theta = A/H$, and $\tan\theta = O/A$. Solve the first two for O and A to obtain $H\sin\theta = O$ and $H\cos\theta = A$. Now substitute into the expression for tangent to obtain $\tan\theta = H\sin\theta/H\cos\theta$, the Hs cancel and presto. Now, this is valid for $0 < \theta < \pi/2$, but we may actually take this as a definition for the tangent function defined for all angles except where cosine vanishes. That's it!

Product-to-Sum Identities

Add (4) and (6) to obtain

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ + \underline{\cos(A-B)} &\underline{= \cos A\cos B + \sin A\sin B} \\ \cos(A+B) + \cos(A-B) &= 2\cos A\cos B \end{aligned}$$

Subtract (4) from (6) to obtain

$$\begin{aligned} \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ - \underline{\cos(A+B)} &\underline{= \cos A\cos B - \sin A\sin B} \\ \cos(A-B) - \cos(A+B) &= 2\sin A\sin B \quad \text{or} \\ -\cos(A+B) + \cos(A-B) &= 2\sin A\sin B \end{aligned}$$

Now add (5) and (7) to obtain

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ + \underline{\sin(A-B)} &\underline{= \sin A\cos B - \cos A\sin B} \\ \sin(A+B) + \sin(A-B) &= 2\sin A\cos B \end{aligned}$$

Thus we have developed, quite easily I might add,

$$\boxed{2\sin A\sin B = -\cos(A+B) + \cos(A-B)} \tag{35}$$

$$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)} \tag{36}$$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)} \tag{37}$$

the product-to-sum identities which, incidentally, we will need when we develop the Euler formulae for Fourier coefficients.

Sum-to-Product Identities

If we set $A+B$ to X and set $A-B$ to Y, then

$$A = \frac{X+Y}{2} \quad \text{and} \quad B = \frac{X-Y}{2}$$

so (36) becomes

$$\boxed{\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}} \tag{38}$$

Under the same substitution, (37) becomes

$$\boxed{\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}} \tag{39}$$

For the tangent function, the process is quite trivial, we simply need to recall (5),

$$\begin{aligned}\tan X + \tan Y &= \frac{\sin X}{\cos X} + \frac{\sin Y}{\cos Y} \\ &= \frac{\sin X \cos Y + \cos X \sin Y}{\cos X \cos Y}\end{aligned}$$

$$\boxed{\tan X + \tan Y = \frac{\sin(X+Y)}{\cos X \cos Y}} \tag{40}$$

Difference-to-Product Identities

Take (38) and replace Y with $-Y$ to obtain

$$\sin X + \sin(-Y) = 2 \sin \frac{X + (-Y)}{2} \cos \frac{X - (-Y)}{2}$$

$$\boxed{\sin X - \sin Y = 2 \sin \frac{X - Y}{2} \cos \frac{X + Y}{2}} \tag{41}$$

Unfortunately, we are not able to take (39) and replace Y with $-Y$ because cosine is even and this approach would be fruitless. We must return to the product to sum identities, specifically, take (35) and make the substitutions

$$X = A + B \quad \text{and} \quad Y = A - B \longrightarrow \quad A = \frac{X+Y}{2} \quad \text{and} \quad B = \frac{X-Y}{2},$$

as before, to obtain

$$2 \sin \frac{X+Y}{2} \sin \frac{X-Y}{2} = -\cos X + \cos Y$$

upon division by -1 we have

$$\boxed{\cos X - \cos Y = -2 \sin \frac{X+Y}{2} \sin \frac{X-Y}{2}} \tag{42}$$

For the tangent function, the process is equally simple as it was for the difference of sines above, replace Y with $-Y$ in (40) to obtain

$$\tan X + \tan(-Y) = \frac{\sin(X + (-Y))}{\cos X \cos(-Y)}$$

$$\boxed{\tan X - \tan Y = \frac{\sin(X-Y)}{\cos X \cos Y}} \tag{43}$$

The Trigonometric Identities Gathered

Reciprocal Identities

$$\begin{aligned}\sec\theta &= \frac{1}{\cos\theta} \\ \csc\theta &= \frac{1}{\sin\theta} \\ \cot\theta &= \frac{1}{\tan\theta}\end{aligned}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Angle-Sum and Angle-Difference Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$
$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\alpha \pm \cot\beta}$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$$

Triple Angle Identities

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$
$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$
$$\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Half Angle Identities

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$
$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$
$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$
$$\cot\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Sum-to-Product Identities

$$\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}$$
$$\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}$$
$$\tan X + \tan Y = \frac{\sin(X+Y)}{\cos X\cos Y}$$

Difference-to-Product Identities

$$\sin X - \sin Y = 2\sin\frac{X-Y}{2}\cos\frac{X+Y}{2}$$
$$\cos X - \cos Y = -2\sin\frac{X+Y}{2}\sin\frac{X-Y}{2}$$
$$\tan X - \tan Y = \frac{\sin(X-Y)}{\cos X\cos Y}$$

Product-to-Sum Identities

$$2\sin A\sin B = -\cos(A+B) + \cos(A-B)$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

Expansion Identities

$$\sin n\theta = \sum_{l=0}^{\infty} (-1)^l \binom{n}{2l+1} \cos^{n-2l-1}\theta \, \sin^{2l+1}\theta$$
$$\cos n\theta = \sum_{k=0}^{\infty} (-1)^k \binom{n}{2k} \cos^{n-2k}\theta \, \sin^{2k}\theta$$
$$\tan n\theta = \frac{\sum_{l=0}^{\infty} (-1)^l \binom{n}{2l+1} \tan^{2l+1}\theta}{\sum_{k=0}^{\infty} (-1)^k \binom{n}{2k} \tan^{2k}\theta}$$

The Derivatives

Behaviour of $\sin\theta/\theta$ about zero

That

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

must firstly be shown. The limit is important in its own right, and it also arises in the proofs of the derivatives of sin and cos. That is the reason for the inclusion of it here. The limit is proven by appeal to the squeeze theorem, a result which, as the name suggests, provides assurance that, if two functions sandwich a third between them, and the two sandwiching functions are approaching the same value in the limit, surely the function squeezed between them must approach the same value. This behaviour is illustrated in figure (3). We state and prove the theorem now for those who do not recall it or who indulge in the intellectual serenity only a complete proof furnishes.

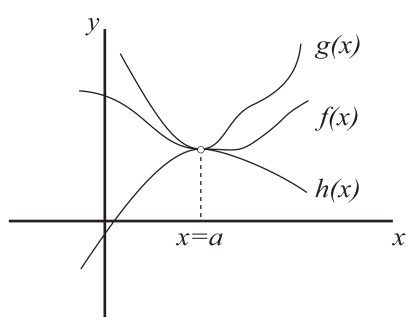

Figure 3: Two functions, g and h, bound a third, f. g and h approach the same limit. Consequently f, sandwiched between the first two, is not able to escape the limiting value of the functions which squeeze it.

Theorem 1. *For functions f, g, and h defined on a neigborhood of $x = a$ except possibly at a, whenever $h(x) \leq f(x) \leq g(x)$ for all $x \neq a$ on the neighborhood and* $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, $\lim_{x\to a} f(x) = L$ *follows.*

Proof. Suppose that, for appropriate $r > 0$, $N_r(a) = \{x \in \mathbb{R} \mid |x - a| < r\}$ is a neighborhood on which $h(x) \leq f(x) \leq g(x)$ except possibly at a. Also suppose $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. We wish to show that $\lim_{x\to a} f(x) = L$. Fix

$\epsilon > 0$. Select $0 < \delta < r$ for which both $\forall x(0 < |x-a| < \delta \longrightarrow |g(x)-L| < \epsilon)$ and $\forall x(0 < |x-a| < \delta \longrightarrow |h(x)-L| < \epsilon)$. Now, for any x for which $0 < |x-a| < \delta$, $|g(x)-L| < \epsilon$ and $|h(x)-L| < \epsilon$ so that $L-\epsilon < g(x) < L+\epsilon$ and $L-\epsilon < h(x) < L+\epsilon$. Because $h(x) \leq f(x) \leq g(x)$ for the same x, we have $L-\epsilon < h(x) \leq f(x)$ and $f(x) \leq g(x) < L+\epsilon$ so that $L-\epsilon < f(x) < L+\epsilon$ or $|f(x)-L| < \epsilon$. Thus, we have proven $\lim_{x\to a} f(x) = L$. □

Alternative Proof. Suppose that, f, g, and h are defined on a neighborhood of a except possibly at a. Also suppose that $h(x) \leq f(x) \leq g(x)$ there. Finally suppose $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. With slightly more sophistication[5] we have

$$L = \lim_{x\to a} h(x) \leq \liminf_{x\to a} h(x) \leq \liminf_{x\to a} f(x) \leq \limsup_{x\to a} f(x) \leq \limsup_{x\to a} g(x) \leq \lim_{x\to a} g(x) = L$$

thus the inequalities are actually equalities, specifically

$$L = \liminf_{x\to a} f(x) = \limsup_{x\to a} f(x)$$

so

$$\lim_{x\to a} f(x) = L\,.$$

□

The squeeze theorem is fantastic because it allows us to infer the behaviour of a function given our knowledge of the behaviour of other related functions. The major drawback in any application of the squeeze theorem is the construction of an appropriate inequality. In the upcoming proof of $\lim_{\theta\to 0} \sin\theta/\theta$, we appeal to a geometric argument in the construction of a fruitful inequality involving $\sin\theta/\theta$. We must firstly revisit some elementary geometry. We

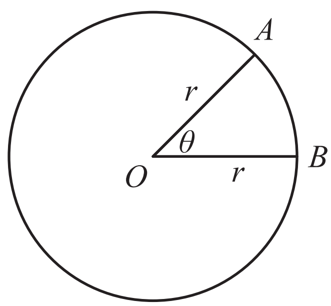

Figure 4: Sector AOB of circle of radius r determined by central angle θ, the sector angle.

will need to know the area of a sector of a circle given its radius and sector angle. For the sector shown in figure (4), we equate the ratios of the area of the sector to the total area of the circle and the measure of the sector angle in radians to the total number of radians in the circle. We have

$$\frac{A_s}{\pi r^2} = \frac{\theta}{2\pi} \text{ or}$$
$$A_s = \frac{r^2\theta}{2}\,.$$

With this result in hand, we explore inequality of areas; consider the unit circle displayed in figure (5). We compare

[5]The limit supremum, lim sup, and the limit infemum, lim inf, have their roots in topological notions of the real line which are beyond the scope of this discussion, but I encourage the motivated reader to learn about some of these things; any decent book about *real analysis* will cover these notions in detail. Readable introductions are available at the library at the section beginning with QA300.

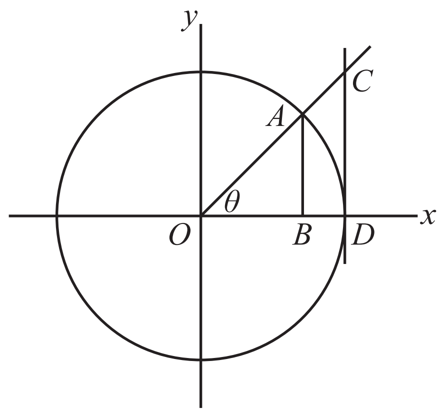

Figure 5: Notice the relationship between the three areas: area of triangle $AOB \leq$ area of sector $AOD \leq$ area of triangle COD, and that this is certainly true when the measure of θ is between 0 and $\pi/2$ radians.

three areas: the area of triangle AOB, sector AOD, and triangle COD. Clearly, the area of triangle AOB is not larger than the area of sector AOD, which, in turn, is not larger than the area of triangle COD. We have

$$\begin{gathered}|\Delta AOB| \leq |\,Sector\;AOD| \leq |\Delta COD| \\ \frac{1}{2}\cos\theta\,\sin\theta \leq \frac{1^2\theta}{2} \leq \frac{1}{2}\cdot 1 \cdot \tan\theta \\ \cos\theta\,\sin\theta \leq \theta \leq \tan\theta\,.\end{gathered}$$

Due to the geometry from which the inequality is obtained, the inequality surely holds when $0 < \theta < \pi/2$. Manipulating the first of the inequalities,

$$\begin{gathered}\cos\theta\,\sin\theta \leq \theta \\ \frac{\sin\theta}{\theta} \leq \frac{1}{\cos\theta}\end{gathered}$$

then the second inequality,

$$\begin{gathered}\theta \leq \tan\theta \\ \theta \leq \frac{\sin\theta}{\cos\theta} \\ \cos\theta \leq \frac{\sin\theta}{\theta}\,.\end{gathered}$$

Thus, we have the inequality

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq \frac{1}{\cos\theta}$$

which remains true on $0 < \theta < \pi/2$. Notice, however, that each expression in the inequality exhibits reflective symmetry through the y axis. Thus, because the inequality holds true from 0 to $\pi/2$, it must hold from $-\pi/2$ to 0 due to the even symmetry of the functions which comprise the inequality. (If this is not clear, sketch the functions.) In summary,

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq \frac{1}{\cos\theta}$$

for θ on $-\pi/2 < \theta < 0$ or $0 < \theta < \pi/2$.

Additionally observe that

$$\lim_{\theta\to 0} \cos\theta = 1 \text{ and } \lim_{\theta\to 0} \frac{1}{\cos\theta} = 1\,.$$

Thus

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

is an immediate consequence of the squeeze theorem.

Derivative of $\sin x$

By the definition of derivative, the limit of the Newton quotient as h approches zero,

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\,,$$

we have the following:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h} \text{ by the angle sum identity for sin} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}
\end{aligned}$$

we have already demonstrated the existence of the limit of the second summand, so we must consider the first summand,

$$\lim_{h\to 0} \frac{\cos h - 1}{h}\,.$$

We multiply numerator and denominator by $\cos h + 1$, this is possible becuase, despite that we introduce zeros in the denominator at $\pi + 2\pi k$ for $k \in \mathbb{Z}$, this does not affect the behaviour at zero. We have

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\cos h - 1}{h} \frac{\cos h + 1}{\cos h + 1} \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\sin^2 h}{h(\cos h + 1)} \text{ by the Pythagorean identity} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1}
\end{aligned}$$

but $\lim_{h\to 0} \sin h/h = 1$ and $\lim_{h\to 0} \sin h/(\cos h + 1) = 0/2 = 0$, so

$$\lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = 1 \cdot 0 = 0\,,$$

and thus

$$\lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}$$

becomes

$$\sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x\,.$$

Thus we have shown that

$$\frac{d}{dx} \sin x = \cos x\,.$$

Derivative of $\cos x$

By a slick application of the chain rule and the fact that sine and cosine curves are out of phase by $\pi/2$ radians,

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin(x + \frac{\pi}{2}) = \cos(x + \frac{\pi}{2}) = -\sin x\,.$$

Booyah!

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

These similarly follow by observing that

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\;\; \cot\theta = \frac{\cos\theta}{\sin\theta},\;\; \tan\theta = \frac{\sin\theta}{\cos\theta},\;\; \sec\theta = \frac{1}{\cos\theta}\,,\;\; \text{and } \csc\theta = \frac{1}{\sin\theta}\,,$$

and slick application of chain rule and quotient rule. We have

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\,,\\
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x\,,\\
\frac{d}{dx}\sec x &= \frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x\tan x, \text{ and}\\
\frac{d}{dx}\csc x &= \frac{d}{dx}(\sin x)^{-1} = -(\sin x)^{-2}(\cos x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\frac{\cos x}{\sin x} = -\csc x\cot x\,.
\end{aligned}$$

Some Anti-derivatives and Integrals

We begin with a summary of the elementary trigonometric derivatives. The development of techniques to obtain anti-derivatives[6] should begin in this way —once we understand the derivatives of these functions, elementary anti-

[6]A brief digression anent terminology is the purpose of this note. A general anti-derivative of a function, say f, is the most general function which has f as its derivative. Some authors employ the locutions 'anti-derivative' and 'general anti-derivative' interchangeably, but they mean different things. Specifically, an anti-derivative of f is any old function with derivative f, but the *general* anti-derivative of f is the most general function with derivative f and, consequently, it has parameters like C in its expression. When I say *the* anti-derivative of f, I refer to the *general* anti-derivative. When I say *an* anti-derivative of f, I refer to any old function with derivative f. I think this use of the definite and indefinite article is a succinct and clear way in which one may disambiguate these notions.

derivatives should follow immediately. Here are the derivatives:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x \quad \text{and} \quad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x \quad \text{and} \quad \frac{d}{dx}\csc x = -\csc x \cot x$$

Incidentally, notice the pattern amongst these derivatives, the derivatives of the co-functions have negatives and they involve the co-functions of the functions involved in the derivatives of the original functions. Also, I remind the reader of the linearity of the anti-derivative,

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx \quad \text{and} \quad \int C\, f(x)\, dx = C \int f(x)\, dx\,.$$

We may state the elementary anti-derivatives, and these should be clear. I encourage you to verify them by differentiation of the anti-derivative to obtain the original integrand. In fact, I encourage you to differentiate every anti-derivative you discover to verify and confirm that your work is correct![7] The following are immediate consequences of the preceding derivatives:

$$\int \cos x = \sin x + C \quad \text{and} \quad \int \sin x = -\cos x + C$$
$$\int \sec^2 x = \tan x + C \quad \text{and} \quad \int \csc^2 x = -\cot x + C$$
$$\int \sec x \tan x = \tan x + C \quad \text{and} \quad \int \csc x \cot x = -\csc x$$

What about anti-derivatives of tan and sec? cot and csc?

$$\int \tan x\, dx = \ln|\sec x| + C \quad \text{and} \quad \int \cot x\, dx = -\ln|\csc x| + C$$
$$\int \sec x\, dx = \ln|\sec x + \tan x| + C \quad \text{and} \quad \int \csc x\, dx = \ln|\csc x - \cot x| + C$$

These are no longer obvious and the demonstration of the truth of these statements requires effort (consequently you should memorize these integrals, they occur with enough frequency to warrant this). The demonstrations of the first

[7] This process, this ability to verify one's work with certainty, is unique to mathematics. This is part of the reason for which this subject is serene, beautiful, and calming! Only in the solace of mathematics is one able to rest one's mind upon the verity of the results! While I write this meloncholic polemic to impell the reader that the study of mathematics is amongst the few true forms of meditation, my mind is drawn to dispel an irksome misconception: that mathematics is dry and that mathematicians are not creative.

Mathematics is pure creation, and mathematical study furnishes the mind with the tools to explore the workings of the universe and reveal its secrets. I am unable to think of a more exciting prospect. I must remind the reader that the entire body of mathematics is a figment of one's mind, no part of it logically rests upon anything real, despite that it is motivated by real problems. This abstraction from reality is the source of its power. The interplay between the two, abstraction and reality, is the source of its excitement. The the greatest mathematicians, whose names we honour by attributing them to important results or entire branches of theory, drew the connections between abstraction and reality.

One thing which is not a misconception is that mathematics is hard and requires a tremendous amount of devotion and effort to master. The difference is that the effort invested to learn mathematics is tuly rewarded, at least intellectually if not practically and financially.

of each pair follow.

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \quad \text{ratio identity for tan}\\ &= -\int \frac{d\cos x}{\cos x} \quad \text{because } \sin x\,dx = -d\cos x\\ &= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln\left|\frac{1}{\cos x}\right| + C\\ &= \ln|\sec x| + C\end{aligned}$$

I must draw the reader's attention to the technique employed in this demonstration, that of the substitution of a trigonometric identity, in this case, a ratio identity for tan. The substitution of a trigonometric identity is the single most fruitful technique in hunting for the anti-derivative of a trigonometric integrand. Now, for the second demonstration, we need a little anti-derivative, $\int 1/(1-x^2)\,dx$.

Consider $\int 1/(1-x^2)\,dx$. Decompose $1/(1-x^2)$ into partial fractions. Partial fraction decomposition is a really important technique for rational functions which are *proper*. A rational function is any quotient of polynomials. Proper rational functions are those in which the degree of the polynomial in the numerator is strictly less than the degree of the polynomial in the denominator. An improper rational function is one in which the degree of the polynomial in the numerator is greater or equal to the degree of the polynomial in the denominator. This language is analogous to that of proper and improper fractions in arithmetic. We have

$$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$$

so $1 = A(1-x)+B(1+x)$ upon multiplication of both sides of the equation by the denominator $(1+x)(1-x)$. Setting x to 1 yields $1 = A(1-1)+B(1+1) = 2B$ so $B = 1/2$. Similarly setting x to -1 yields $1 = A(1-(-1))+B(1+(-1)) = 2A$ so $A = 1/2$. Thus,

$$\begin{aligned}\int \frac{1}{1-x^2}\,dx &= \int \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}\,dx = \frac{1}{2}\left(\int \frac{1}{1+x} + \frac{1}{1-x}\,dx\right)\\ &= \frac{1}{2}\left(\ln|1+x| + \ln|1-x|\right) + C = \frac{1}{2}\ln\frac{|1+x|}{|1-x|} + C\\ &= \frac{1}{2}\ln\left(\frac{|1+x|}{|1-x|}\frac{|1+x|}{|1+x|}\right) + C\end{aligned}$$

This last multiplication is permitted despite that it eliminates the point -1 from the domain, it wasn't in the domain of the original integrand either. Continuing, we have

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\frac{(1+x)^2}{|1-x^2|} + C\,.$$

Now, we consider $\int \sec x\,dx$.

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{1}{\cos x}\frac{\cos x}{\cos x}\,dx$$

Again, this last multiplication does not change the domain of the integrand, so it is permitted. We continue,

$$\int \sec x\, dx = \int \frac{\cos x}{\cos^2 x}\, dx$$
$$= \int \frac{\cos x}{1-\sin^2 x}\, dx = \int \frac{d\sin x}{1-\sin^2 x}$$

but this last integral is simply the little integral we demonstrated above with $\sin x$ substituted for x. Thus

$$\begin{aligned}\int \sec x\, dx &= \frac{1}{2}\ln\frac{(1+\sin x)^2}{|1-\sin^2 x|} + C = \frac{1}{2}\ln\frac{(1+\sin x)^2}{|\cos^2 x|} + C \\ &= \frac{1}{2}\ln\frac{(1+\sin x)^2}{\cos^2 x} + C = \frac{1}{2}\ln\left(\frac{1+\sin x}{\cos x}\right)^2 + C = \ln\left(\left(\frac{1+\sin x}{\cos x}\right)^2\right)^{1/2} + C \\ &= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C \\ &= \ln|\sec x + \tan x| + C\,.\end{aligned}$$

Orthogonality Relations

In this section, $n \in \mathbb{Z}_0^+$ and $m \in \mathbb{Z}_0^+$, in other words, m and n are non-negative integers.[8]

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx = 0 \quad \text{for all } m, n \in \mathbb{Z} \tag{44}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\, dx = \begin{cases} 0, & \text{when } n \neq m \\ 2L, & \text{when } n = m = 0 \\ L, & \text{when } n = m \neq 0 \end{cases} \tag{45}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & \text{when } n \neq m \\ 0, & \text{when } n = m = 0 \\ L, & \text{when } n = m \neq 0 \end{cases} \tag{46}$$

Why? Before we proceed, this is an ideal place to inject further properties which are consequences of various symmetries.

Proposition 3. *The average value of the pure sinusoid $A\sin(\omega t + \phi)$ over one period is zero.*

[8] $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers. Some authors include 0 amongst the natural numbers for convenience, they are justified and they will be clear, but you need to be aware of this. I nevertheless prefer the notation $\mathbb{N}_0$ for the set of naturals including 0, it is explicit. Unambiguously, we also have $\mathbb{Z}$ for the set of integers, $\mathbb{Z}^+$ for the positive integers (this coincides with $\mathbb{N}$), $\mathbb{Z}^-$ for negative integers, $\mathbb{Z}_0^+$ for the non-negative ingegers, and $\mathbb{Z}_0^-$ for the non-positive integers. The 'Z' is employed because 'zahl' is the german word for 'number'. The set of real numbers is denoted by $\mathbb{R}$, which is partitioned into the rationals, $\mathbb{Q}$, and the irrationals, $\mathbb{Q}^{\complement}$. In other words, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\complement}$, but $\mathbb{Q} \cap \mathbb{Q}^{\complement} = \varnothing$. Finally, $\mathbb{C}$ denotes the set of complex numbers, $\{a + ib \mid a, b \in \mathbb{R}\}$...but you already knew all of this. If you be interested in the number systems, I recommend that you read Paul Halmos's classic *Naïve Set Theory* followed by Edmund Landau's classic *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers*. These books will blow your mind.

Proof. The period of $A \sin(\omega t + \phi)$ is $2\pi/\omega$. Consider $\int_\tau^{\tau+2\pi/\omega} A \sin(\omega t + \phi)\ dx$, the integral of the pure sinusoid over one period. We have

$$\begin{aligned}
\int_\tau^{\tau+2\pi/\omega} A \sin(\omega t + \phi)\ dx &= -\frac{A}{\omega} \cos(\omega t + \phi) \Big|_\tau^{\tau+2\pi/\omega} \\
&= -\frac{A}{\omega} \left(\cos\left(\omega \left(\tau + \frac{2\pi}{\omega} \right) + \phi \right) - \cos(\omega(\tau) + \phi) \right) \\
&= -\frac{A}{\omega} \left(\cos(\omega\tau + 2\pi + \phi) - \cos(\omega\tau + \phi) \right) = -\frac{A}{\omega} \left(\cos(\omega\tau + \phi) - \cos(\omega\tau + \phi) \right) \\
&= 0
\end{aligned}$$

and we have proven the proposition. Incidentally, the integral over any whole multiple of a period is also zero. □

Proposition 4. *The integral of an even function over an interval symmetric about the origin is twice the integral over half of the interval. The integral of an odd function over an interval symmetric about the origin is zero. Specifically, when f is even,*

$$\int_{-a}^{a} f(x)\, dx = 2 \int_0^a f(x)\, dx\,,$$

and, when f is odd,

$$\int_{-a}^{a} f(x)\, dx = 0\,.$$

Proof. When f is even, consider $\int_{-a}^{a} f(x)\, dx$,

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx\,.$$

Now, in the first integral, let $u = -x$ then $dx = -du$, and $\int_{-a}^{0} f(x)\, dx$ becomes $-\int_a^0 f(-u)\, du = \int_0^a f(-u)\, du$, but f is even, so this is simply $\int_0^a f(u)\, du$. Thus

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = \int_0^a f(u)\, du + \int_0^a f(x)\, dx = 2 \int_0^a f(x)\, dx\,.$$

For the second claim, the proof of which is similar to the previous, suppose that f is odd and consider $\int_{-a}^{a} f(x)\, dx$,

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx\,.$$

Now, in the first integral, let $u = -x$ then $dx = -du$, and $\int_{-a}^{0} f(x)\, dx$ becomes $-\int_a^0 f(-u)\, du = \int_0^a f(-u)\, du$, but f is odd, so this is simply $\int_0^a -f(u)\, du$. Thus

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = \int_0^a -f(u)\, du + \int_0^a f(x)\, dx = 0.$$

□

Now, let's examine relation (44). Because cosine is even and sine is odd, by proposition 1, $\sin x \cos x$ is odd. Now, by proposition (4), because we are integrating over $[-L, L]$, a symmetric interval about the origin, the integral in (44) must be zero. For the remaining demonstrations, for equations (45) and (45), we must employ the product to sum identities. In the product to sum identities, (37) and (35), with A replaced with $n\pi x/L$ and B replaced with $m\pi x/L$, we have

$$\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \frac{1}{2}\left(\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right) \quad \text{and} \tag{47}$$

$$\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \frac{1}{2}\left(-\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right). \tag{48}$$

When m and n are not equal, we are integrating the sinusoids in (47) and in (48) over integer multiples of their periods and the average value of pure sinusoids is zero, thus these integrals are zero when m and n are not equal. When m and n are both zero, (47) becomes 1 and $\int_{-L}^{L} dx = 2L$. When m and n are both zero, (48) becomes 0, the integral of which is clearly zero. Finally, when $m = n \neq 0$, (47) and (48)become half of the sum of a pure sinusoid and 1, the integrals of which are L.

Fourier Series

We develop Fourier Series[9] purely formally in this section. A Fourier Series for the function $f(x)$ has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{49}$$

in which the coefficients a_n ($n \in \mathbb{N}_0$) and b_n ($n \in \mathbb{N}$) are given by the Euler Formulae for Fourier Coefficients,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx\,, \tag{50}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n \in \mathbb{N}, \text{ and} \tag{51}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n \in \mathbb{N}. \tag{52}$$

[9]Jean Baptiste Joseph Fourier (1768 – 1830) was a French mathematician and physicist to whom we attribute the trigonometric series which bears his name. The series was known by mathematicians of his day, but he employed it aggressively and successfully answered questions in his studies of heat conduction. The Fourier transform is also named in his honour. Incidentally, Fourier is acknowledged to have discovered what we call the greenhouse effect. Hot!

Why? The orthogonality relations developed in the previous section come to the rescue! Take the series (49) and integrate over $[-L, L]$. We have, assuming that we are able to integrate the series termwise[10]

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L} \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \int_{-L}^{L} \frac{a_0}{2}\,dx + \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\,dx + b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\,dx\end{aligned}$$

but each of the integrals on the left is zero for each $n \in \mathbb{N}$ because the average values of these sinusoids over whole multiples of their periods are zero. Thus we have

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2}\,dx = \frac{a_0}{2}x\bigg|_{-L}^{L} = \frac{a_0}{2}(L - (-L)) = \frac{a_0}{2}(2L) = a_0 L$$

Thus, upon division of both sides by L, we have (50). Now we apply a similar approach, we multiply both sides of (49) successively by $\cos\left(\frac{m\pi x}{L}\right)$ and by $\sin\left(\frac{m\pi x}{L}\right)$ each time integrating over $[-L, L]$ and employ the orthogonality relations to discover any simplification. We have, upon multiplication of both sides of (49) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating,

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx &= \int_{-L}^{L} \frac{a_0}{2}\cos\left(\frac{m\pi x}{L}\right) + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx \\ &= \frac{a_0}{2}\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx + b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\end{aligned}$$

but, because the average value over a period of the sinusoid is zero, the first integral vanishes, and, by the orthogonality relations, the first integral in the sum vanishes for all values of n except $m = n$ and the second integral vanishes, so we have

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = a_m \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = a_m L\,.$$

Thus, upon division of both sides by L, we have (51). Now, for the demonstration of (52), we have, upon multiplication of both sides of (49) by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating,

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx &= \int_{-L}^{L} \frac{a_0}{2}\sin\left(\frac{m\pi x}{L}\right) + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx \\ &= \frac{a_0}{2}\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)dx + \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx + b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\end{aligned}$$

but, similar arguments allow us to eliminate the first integral on the right and the first integral in the sum on the right, so we have

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = b_m \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = b_m L\,.$$

[10] This is a formal derivation in which we are not concerned with convergence. For these issues of rigour, explore the notions of uniform convergence and norm convergence and other fun things in topology, functional analysis, and measure theory. Fourier opened a can of worms with various claims he had made which were eventually demonstrated to be false in general. The bulk of 19^{th} century mathematics was devoted to the rigorizing of these claims, and the bulk of 20^{th} century mathematics was devoted to the generalizations of those results.

Thus, upon division of both sides by L, we have (52). We now state without proof a *Fourier theorem*, a theorem which furnishes the conditions under which a Fourier series converges.[11]

Theorem 2. *For a function f, piecewise continuous on $[-L, L]$ and periodic with period $2L$ on the entire x axis, its series, (49), with coefficients given by (50), (51), and (52), converges to the mean value*

$$\frac{f(x^+) + f(x^-)}{2}$$

of the one-sided limits of f at each point $x \in \mathbb{R}$ at which the one-sided derivatives $f'(x^+)$ and $f'(x^-)$ exist.[12]

The Basel Problem

The Basel Problem, a famous problem in analysis named for the city in northern Switzerland, the birthplace of Euler and the Bernoullis, was first posed by Pietro Mengoli in 1644. Leonhard Euler was introduced to this problem by the Bernoullis who unsuccessfully attempted it. The problem asks for the exact value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and a rigorous proof of the value. Euler had discovered that

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

in 1734, but had no rigorous justification. His arguments were based on manipulations not justifiable at the time. Euler produced a fully rigorous proof in 1741. We present one of many possible proofs, not Euler's, which requires too much machinery at this point. The one we demonstrate employs Fourier series in a clever way; this is a non-trivial application of Fourier series. All we really do is find the Fourier series of x^2 and evaluate it at a certain point. That's it! Here we go. Wait, one final remark, throughout this section we employ the fact that $\cos n\pi = (-1)^n$. Here we go! Construct the function $f(x)$ by repeating the behaviour of x^2 on the interval $[-\pi, \pi]$ to obtain its even periodic extension shown in figure 6. $f(x)$ is even, so $f(x)$ multiplied by sin is odd, thus the integrals involving sine vanish; the series for $f(x)$ is a cosine series. We compute the coefficients now.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx \\ &= \frac{1}{\pi} \frac{x^3}{3} \bigg|_{-\pi}^{\pi} = \frac{1}{3\pi}(\pi^3 - (-\pi)^3) = \frac{1}{3\pi}(\pi^3 + \pi^3) = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3} \end{aligned}$$

[11] See *Fourier Series and Boundary Value Problems* by Brown and Churchill. It is a clear and readable presentation of Fourier series and applications to partial differential equations suitable for the mathematically inclined student of engineering. Don't recall it from the library though because you'll probably get the copy I have borrowed and that would mildly inconvenience me.

[12] The right hand derivative of f at x_0 is defined by

$$f'({x_0}^+) = \lim_{x \to {x_0}^+} \frac{f(x) - f({x_0}^+)}{x - x_0}.$$

Similarly, the left hand derivative is defined by

$$f'({x_0}^-) = \lim_{x \to {x_0}^-} \frac{f(x) - f({x_0}^-)}{x - x_0}.$$

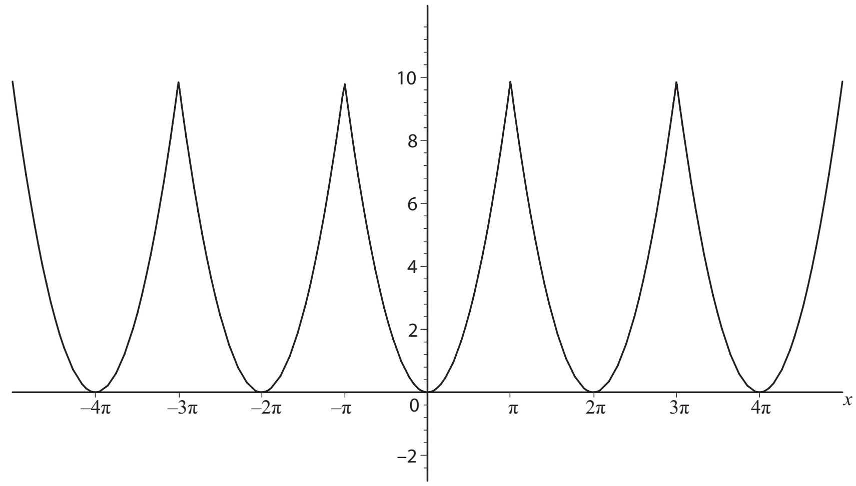

Figure 6: Plot of the function $f(x)$, the even periodic extension of x^2 with period 2π.

and

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx \\
&= \frac{1}{\pi}\left(\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right)_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2\sin(n\pi)}{n} + \frac{2\pi\cos(n\pi)}{n^2} - \frac{2\sin(n\pi)}{n^3}\right) - \frac{1}{\pi}\left(\frac{(-\pi)^2\sin(n(-\pi))}{n} + \frac{2(-\pi)\cos(n(-\pi))}{n^2} - \frac{2\sin(n(-\pi))}{n^3}\right) \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}
$$

Thus the Fourier series for $f(x)$,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)\,,$$

converges to f for all $x \in \mathbb{R}$. Now, when $x = \pi$, we have

$$\begin{aligned}
f(\pi) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi) \\
\pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi) \\
\pi^2 - \frac{\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n \\
\frac{2\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \\
\frac{2\pi^2}{4 \cdot 3} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

Booyah!